Summer 6-28-2010

Incorporating the History of Mathematics into Secondary Curriculum

Lacy Gaines

INCORPORATING THE HISTORY OF MATHEMATICS INTO THE SECONDARY CURRICULUM

by

LACY JAE GAINES

Submitted to the Honors College
of Texas A&M University-Commerce
in partial fulfillment of the requirements
for the degree of
BACHELORS OF SCIENCE
July 28, 2010

INCORPORATING THE HISTORY OF MATHEMATICS INTO THE SECONDARY CURRICULUM

Approved:

____

____

____

____

ABSTRACT

INCORPORATING THE HISTORY OF MATHEMATICS INTO THE SECONDARY CURRICULUM

Lacy Jae Gaines
Texas A&M University-Commerce, 2010

Adviser: Pamela S. Webster, Ed.D.

The purpose of this study was to investigate how the history of mathematics can be integrated into the high school mathematics curriculum and its effectiveness. The researcher examined the high school subjects of algebra, geometry, pre-calculus, and calculus and how their historical significances can be applied to enhance students' understanding and motivation. The paper includes historical examples and a lesson plan for each of these courses. In addition, the study consisted of the researcher presenting a lesson over the Pythagorean Theorem, its history, and some proofs of the theorem to a college mathematics class for future elementary teachers. An attitudinal survey was given before and after the lesson, and an assessment was given after the lesson. The lesson included a PowerPoint over the history of the Pythagorean Theorem, two Pythagorean Theorem puzzles, and the proofs by Bhaskara and President James Garfield. The results of the study were that the majority of the students increased their ratings after the lesson and that students did well on the assessment. The researcher believes that the lesson using the theorem's history helped students understand the Pythagorean Theorem, learn where the theorem came from, and realize the purpose of the theorem.

Table of Contents

Introduction

The thesis paper discusses whether or not incorporating the history of mathematics into high school classrooms helps students understand concepts in math and will assist in their appreciation of the need for and purpose of math. It is the researcher's belief that many students take math classes and never realize the purpose of math; they never understand that the historical figures in mathematics developed and studied math to solve problems in their society. This paper also discusses the major contributions of historical mathematicians and how to integrate their work and theories into the current secondary mathematics curriculum in public schools. In particular, the study focuses on the secondary math subjects of algebra, geometry, pre-calculus, and calculus. The author includes examples of how the history of math is beneficial in the subjects, the history of the development of the subjects, and a lesson plan for each subject that involves integrating the history of the subject. Also, a study along with the results of the study is included during which the researcher taught a lesson using the history of mathematics and students were given attitudinal surveys.

Review of Related Literature

The Principles and Standards for School Mathematics, created by the National Council of Teachers of Mathematics, describes a vision, principles, and standards for school mathematics for prekindergarten through 12th grade. Six principles are described for successful school mathematics – equity, curriculum, teaching, learning, assessment, and technology (NCTM, 2000, p. 11). These themes are important for teachers and administrators to create the best mathematics environment possible, and the standards in this book discuss the mathematical expectations of high school students. The author uses these standards in the thesis to show the connection between the history of math and current requirements of high school math students. Throughout this paper, the principles and standards are referred to in order to show how the integration of the history of math into the classroom supports these principles and standards.

The learning principle states, "Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge" (NCTM, 2000, p. 20). This principle is demonstrated in "A Mathematical History Tour" by Loretta Kelley from *The Mathematics Teacher*, which discusses the importance of the history of math in a classroom. Kelley (2000) states three reasons for incorporating history of math into lessons – "(1) to put mathematics in its historical context; (2) to show the interaction of mathematics with the cultures of which it has been a part; and (3) to shed light on the learning – and therefore teaching – of mathematics" (p. 14). Teachers and students are always trying to find ways to remember certain steps to work out a problem. The importance of putting math in its historical context is that students remember mathematical concepts more when they can associate it to names, places, or stories (Kelley, 2000, p. 14). For example, telling students the story of

Archimedes realizing the concept of density while he was sitting in the bath tub and then running through the streets shouting "Eureka! (I have found it)" will entertain them and will help them remember density and how it was discovered. "To break the image of mathematics as a 'boring' subject we can add color and enliven it by considering its human side and from time to time expose students to anecdotes from the lives of mathematicians" (Avital, 1995, p. 11).

"A Mathematical History Tour" also supports the curriculum principle, which declares that a math topic can be considered important if it expands students' acknowledgement of math as a human creation (NCTM, 2000, p. 15). The history of math shows the interaction of cultures; for example, Kelley (2000) discusses how the Chinese used mathematics primarily to solve practical problems such as building walls, farming, and dividing estates. The Greeks, on the other hand, did not study math for practical purposes, focusing more on the why, not how to apply it. History can also help us teach; according to Kelley (2000), "Parallels are observed between the historical development of mathematical concepts and the way that they develop in an individual mind" (p. 16). It is crucial for teachers to get to know the students' thinking process and make sure their process and prerequisite knowledge is correct. This notion promotes the teaching principle which asserts that successful teaching demands understanding mathematics, students as learners, and didactic strategies (NCTM, 2000, p. 17).

Further evidence of current research that supports the teaching principle is seen in "The Role of History in a Mathematics Class" by Gerald L. Marshall and Beverly S. Rich. This article examines several research studies done on the effects of implementing the history of math into math courses. Marshall and Rich discuss a study done by Jardine in 1997 in which West Point students learn math by studying the history of math in calculus classes; they had to write a paper on an historical mathematician and give a presentation to the class on him or her (Jardine, 1997, p. 115). This is a great way for students to get introduced to mathematicians and math concepts. At the end of the class, the students of West Point took a survey and said learning about the history stimulated them to learn the math and improved their experience in the class (Marshall & Rich, 2000, p. 704). Doing this activity in a class can help initiate interest for students and make learning more enjoyable. This type of activity emphasizes the teaching principle because well-chosen tasks encourage students' curiosity and draw them into math (NCTM, 2000, p. 17). This article also describes a study in which Furinghetti (1997) used case studies with four teachers examining connections between math history and math education. Furinghetti's (1997) study concluded that "history in mathematics facilitates understanding through reflection and enhances mental imaging by expanding a student's conception of mathematics" (p. 55).

"The Effects of History of Mathematics on Attitudes toward Mathematics of College Algebra Students" by McBride and Rollins (1977) involves a study done on college algebra students. It seems today as if many students have poor attitudes toward math, thinking it is too difficult or it is useless. In this study, McBride and Rollins (1977) examine two college algebra classes with two instructors around the same age and years in experience. One instructor taught using history of math and the other without using the history, and both instructors used the same textbook without history (p. 57). The purpose of this study was to see whether or not including history of math in the class affected students' attitudes toward math because students' attitudes in math tend to decline after middle school. "The primary conclusion of the study is that the program using items from the history of mathematics was effective as a means of promoting positive student attitudes toward mathematics" (McBride & Rollins, 1977, p. 60). The author of this thesis paper thinks this finding is important because if teachers can present the material in a fun and intriguing manner, students will have a better attitude and will be more successful.

The learning principle also states that "students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge" (NCTM, 2000, p. 20). This means that students will learn better if the teacher is able to integrate the new material with something the students already know or understand. This principle is supported in "Who? How? What? A Strategy for Using History to Teach Mathematics" by Wilson and Chauvot (2000) that describes the benefits of the history of math in a math classroom. Wilson and Chauvot (2000) discuss how using history of math does not have to be extra work for teachers but a tool for effective teaching. It may turn out to be less work if teachers spend a few moments in class teaching the history of the concept, how the concept was developed, and how the history connects to the new material because students may thoroughly understand it more quickly and remember it better. The authors of the article believe this practice produces four essential benefits – "sharpens problem-solving skills, lays a foundation for better understanding, helps students make mathematical connections, and highlights the interaction between mathematics and society" (Wilson & Chauvot, 2000, p. 642).

Using history emphasizes why certain things are done in mathematics, and the reason behind the practice helps students' comprehension. There are many connections between math and society: the Egyptians' need for the pyramids caused the development of complicated mathematics; the space race caused a need for greater research in math; and some societies' beliefs and even superstitions caused profound mathematicians to delay their work in math. "The ability to make mathematical connections has been emphasized by teachers, employers, and the NCTM's Principles and Standards for School Mathematics as one of the most important goals of mathematics learning" (Wilson & Chauvot, 2000, p. 642). Wilson and Chauvot (2000) also give a method for incorporating the history of math into lessons – to address "who does mathematics, how mathematics is done, and what mathematics is" (p. 643). By answering these questions, students will learn about different cultures, realize that they have the potential to do great work, just as the historical figures they learn about, learn all the different methods these historical figures used, and understand how humans developed math over centuries. For instance, students will realize that a concept or formula can be understood in a short period of time, but mathematicians may have taken hundreds of years to discover that formula.

According to the NCTM (2000), the teaching principle states, "In effective teaching, worthwhile mathematical tasks are used to introduce important mathematical ideas and to engage and challenge students intellectually" (p. 18). This can be achieved by giving students specific problems from throughout the history of mathematics. In "Using Problems from the History of Mathematics in Classroom Instruction," Swetz (1995a) states that historical problems provide case studies of many current topics students face today (p. 25). Historical problems can also clarify concepts being taught. For example, ancient Babylonian problems can be referenced when discussing the method of completing the square (Swetz, 1995a, p. 28).

The teaching principle states that teachers need to know what students often have difficulty with and ways to help bridge common misunderstandings (NCTM, 2000, p. 17). In "History of Mathematics Can Help Improve Instruction and Learning," Avital (1995) believes that a knowledge and understanding of the historical development of mathematics can contribute in attaining insight into students' learning difficulties (p. 3). Investigating difficult mathematical concepts throughout history can help teachers understand complexities students face. Some of these difficulties include the acceptance of negative numbers, the use of symbolic notation, and rigor and abstraction (Avital, 1995, p. 4). Studying the history of math can help teachers teach these intricate concepts. For example, Avital (1995) claims that generalizations given before examples can become lost, which is the method used today in most secondary school textbooks, but historical mathematical developments went from specific examples to generalizations (p. 6).

According to the curriculum principle, mathematics is composed of many interconnections between subjects such as algebra and geometry, and it is important for these interconnections to be displayed in the curriculum, materials, and lessons (NCTM, 2000, p. 15). The history of math is a portal to realizing all the connections within the subjects of algebra, geometry, pre-calculus, and calculus. In *A History of Mathematics*, Carl Boyer gives an in-depth history of math starting with the origins of numbers and counting and the Egyptians and continuing all the way through the twentieth century. Boyer also describes the most prominent mathematical figures in history, such as Pythagoras, Euclid, Archimedes, Descartes, Newton, and Leibniz. This book is especially important in the portion of this study where the researcher will give specific examples of how the history of math is beneficial in the subjects of algebra, geometry, pre-calculus, and calculus. For example, next in this thesis, the author provides examples for each of the subjects. The researcher presents a general history, and then focuses on the mathematicians who developed important mathematical aspects of the subject. Then, the author discusses the history will benefit students and how it can be implemented into a lesson and possible activities for a class. Finally, the researcher uses the *Principles and Standards for School Mathematics* and the Texas Essential Knowledge and Skills (TEKS): Math Grades 9-12 to describe the material being taught and its requirements in the secondary classroom by providing a lesson plan.

Incorporating the history of math into the classroom shows students where math comes from, why it was created, and helps them make connections. Learning about the history of math creates a deeper appreciation for the subject and its contributors and helps students realize the long process required to develop a concept that they may learn in one class period. Teaching history of math in the high school curriculum allows our students to appreciate math as a human creation. The history of math can be implemented effectively into the secondary education curriculum (i.e. the calculus example). This thesis paper includes more research that supports the integration of history of math into math classrooms and more specific examples and lesson plans in algebra, geometry, pre-calculus, and calculus, including how the study of the history of math will benefit students.

Example for Algebra

Some important topics in high school algebra include solving quadratic equations by various methods, complex numbers, and irrational numbers. Also, many topics in algebra are connected with other subjects such as geometry. Many cultures and mathematicians have contributed to the history of algebra, including the Babylonians, Egyptians, Chinese, Greeks, Euclid, Diophantus, al-Khowârizmî, and Vieta. As stated earlier, there are parallels between the historical developments of mathematical concepts and how they develop in the mind (Kelley, 2000, p. 16). Some number systems were developed by trying to solve certain types of equations. Students' understanding of the mathematical development of number systems should be a basis for their work in finding solutions for certain types of equations (NCTM, 2000, p. 291). The author will investigate the history of these topics in algebra, discuss how it would be beneficial if incorporated into a high school algebra class, and determine how it can be done so.

The Babylonians had several purposes for developing algebra which included solving many of society's problems. Much of what we know about their mathematics comes from clay tablets where many calculations were written, such as tables calculating squares and cubes and solving quadratic equations, which is an important piece of high school algebra. According to Katz (2007):

> Mesopotamian mathematics (often called Babylonian mathematics) had two roots – one is accountancy problems, which from the beginning were an important part of the bureaucratic system of the earliest Mesopotamian dynasties, and the second is a "cut and paste" geometry probably developed by surveyors as they figured out ways to understand the division of the land (p. 186-187).

This shows students a purpose of the development of algebra, and teachers can use the "cut and paste" geometry to demonstrate concepts such as factoring, which can be represented as displaying the quadratic equation as the area of a rectangle and the dimensions of the rectangle are the equation's factors. Students sometimes do not truly understand what factoring is; often students factor an equation using various methods and then think they are not finished and try to multiply the factors back out.

Everything the Babylonians accomplished mathematically was rhetorical, meaning it was all written out word for word without the use of symbols. For example, one of their problems would be, "I have added the area and two-thirds of the side of my square and it is 0;35. What is the side of my square?" which in modern notation would be expressed as $x^2 + \frac{2}{3}x = \frac{35}{60}$ (Burton, 2007, p. 65). The rhetorical stage would remain until Diophantus developed syncopated algebra. According to the NCTM's Algebra Standard for Grades 9-12, students are expected to "use symbolic algebra to represent and explain mathematical relationships" (p. 296). Teachers can give students rhetorical problems from ancient Babylonia to be translated into symbolic algebra and then solve.

The Texas Essential Knowledge and Skills for Mathematics (2009) states, "The student understands there is more than one way to solve a quadratic equation and solves them using appropriate methods" for Algebra I (p. 3). The Babylonians would solve quadratic equations using the following formula written in modern notation: $x = \sqrt{(\frac{a}{2})^2 + b} - \frac{a}{2}$, which is completing the square, which led to the derivation of the quadratic formula. The quadratic formula is extremely important in high school algebra. The Babylonians would only solve quadratic equations of the form $x^2 + px = q, x^2 = px + q$, and $x^2 + q = px$ because they only dealt with positive roots. Building on the work of the Babylonians that takes us to the Arabs, Al-Khwârizmî was an Arab mathematician, and through his works on arithmetic and algebra, Europe became familiar with Hindu numerals and algebraic approaches (Burton, 2007, p. 240). Al- Khwârizmî would represent equations geometrically and solve by what is known today as completing the square. One teacher used his approach and felt for the first time that students truly understood the solution by completing the square (Avital, 1995, p. 8). Solving quadratic equations was a very long process in which mathematicians and cultures built on the works of others. Studying the history gives opportunities for appreciation. Sometimes students think math is all about rules; they often believe there is only one way to solve something. However, the researcher believes that investigating the history of quadratic equations shows the many ways to solve them and math's flexibility.

Another high school algebra topic that has a long history of its discovery is the notion of complex numbers. Students are expected to understand complex numbers as solutions to quadratic equations that do not have real solutions (NCTM, 2000, p. 290). Early mathematicians did not accept negative numbers so they were limited to only solving quadratics with positive solutions. Burton (2007) states Cardan was the first person to take notice of negative roots even though he called them "fictitious" and another feature in his work, the *Ars Magna,* was the recognition of imaginary numbers (p. 323). He came across these numbers when he tried to divide 10 into 2 parts with a remainder of 40, resulting in the roots $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$. Approximately fifteen years after Cardan's recognition of negative roots, imaginary numbers were accepted by other mathematicians. "Bombelli was the first mathematician bold enough to accept the existence of imaginary numbers, and hence to throw some light on the puzzle of irreducible cubic equations" (Burton, 2007, p. 327). In the seventeenth century, René Descartes labeled an expression like $\sqrt{-1}$ as imaginary numbers, and in the 1700s, Euler developed the notation for imaginary numbers: $\sqrt{-1} = i$ and $-1 = i^2$. The study of imaginary numbers shows that mathematical discoveries are sometimes made for real-world purposes or are made out of practicing math purely, but the concepts from pure mathematics become applicable to many fields. For example, complex numbers are used in many advanced fields such as electrical engineering and quantum mechanics, and a strong mathematical foundation is necessary for these fields and upper level courses such as differential equations.

Irrational numbers can also be solutions to equations. Like imaginary numbers, irrational numbers have a long history of being established and were first seen by members of Pythagoras' school. Pythagoras is one of the great heroes of Greek mathematics. His followers believed that everything is number, and numbers represented love, gender, and hate (Lewinter & Widulski, 2002, p. 46). Pythagoras strongly believed in whole numbers and the ratio of whole numbers (rational numbers). It must have been devastating when they established that not all numbers can be written as a ratio of integers, but it is not known who first found this or how it was done (Burton, 2007, p. 111). The discovery of irrational numbers occurred after Pythagoras' death so his followers vowed to keep their discovery a secret since it discredited everything they believed in. Legend has it that Hippasus was thrown overboard at sea for revealing their secret (Smith, 1996, p. 23). When introducing irrational numbers, teachers can show students Aristotle's indirect proof of irrational numbers to demonstrate how a diagonal and side of length one unit of a square are not commensurable. From his proof, the concept of incommensurability can be understood. According to Livio (2002):

> Before this discovery, mathematicians had assumed that if you have any two line segments, one of which is longer than the other, then you can always find some smaller unit of measure so that the lengths of both segments will be exact whole-number multiples of this smaller unit (p. 39).

By studying the history of irrational numbers, hopefully students will receive a true understanding of what irrationality actually means.

"Students continually use problem-solving, language and communication, and reasoning (justification and proof) to make connections within and outside mathematics" (TEKS, 2009, p. 1). By studying the history of Greek algebra, we see the introduction to geometric algebra and a movement towards a more abstraction based on proofs. Constructions by compass and ruler allowed the Greeks to prove algebraic identities geometrically. For example, the Greeks proved the identity $(a+b)^2 = a^2 + b^2 + 2ab$ geometrically, shown in Figure 1 (Bashmakova & Smirnova, 2000, p. 16).

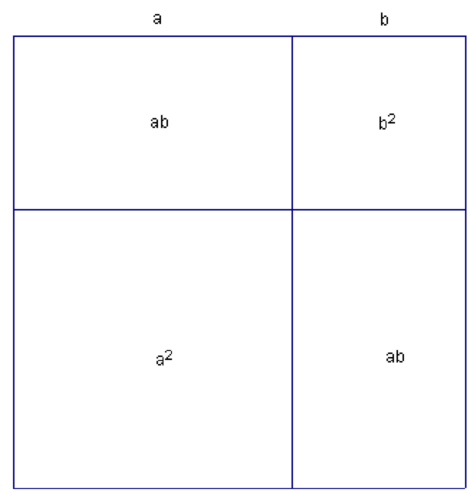


Figure 1

According to the algebra standard for grades 9-12, all students should “represent and analyze mathematical situations and structures using algebraic symbols” (NCTM, 2000, p. 296). The identity given above is an algebraic explanation for Figure 1, which supports this standard. Students will have a visual justification for the algebra they have computed. This visual representation enhances students’ understanding, which supports the learning principle because they are using prior experiences and knowledge.

The ultimate use of history in algebra is to show students the lengthy amount of time it took to develop concepts such as solving equations and imaginary numbers. Investigating the history of algebra concepts shows their purposes in society and why they were developed. Students may be comforted by the fact that it took so long for mathematicians to understand and discover these concepts if students do not understand quickly. The researcher has devised a sample lesson plan that integrates the history of algebra. The lesson plan explores al-

Khwarizmi's completing the square method to solve quadratic equations (see Appendix A: Algebra Lesson).

Example for Geometry

Geometry is said to have originated in ancient Egypt in order to determine how much land was gained or lost due to the floods of the Nile River. The word geometry actually comes from the Greek words meaning “earth” and “measure” (Burton, 2007, p. 53). Pi, the Pythagorean Theorem, and lengths, areas, and volumes are important aspects of geometry, and this paper examines their histories and how to apply them in the classroom.

Pi, the ratio of the circumference and the diameter of a circle, is one of the most used mathematical constants in geometry to find areas and volumes of many shapes. Students are expected to “find areas of regular polygons, circles, and composite figures” (TEKS, 2009, p. 9). Many students endure their math classes without understanding what $\boldsymbol{\pi}$ really is while other students take $\boldsymbol{\pi}$ for granted. Investigating the lengthy history of $\boldsymbol{\pi}$ will uncover $\boldsymbol{\pi}$’s development as a human creation and its many purposes not only in math but also in science and engineering.

In a problem of the Rhind Papyrus, the scribes’ procedure for finding the area of a circle was $A = (d - \frac{d}{9})^2$ and setting this to the modern area formula gives $\pi = 3.1605 \ldots$ The Babylonians’ method of finding circumference was found by taking three times the diameter, which gives their pi value of 3 (Burton, 2007, p. 55). The Egyptian mathematics is found in constructions in some passages of the Bible. Problems pertaining to these ancient constructions can be given to students to show math’s purposes during those times and can apply to other areas of math such as having to compute conversions. In *The Measurement of a Circle*, Archimedes gave an approximation of $\boldsymbol{\pi}$, and according to Burton (2007), Archimedes’ approximation was based on the following fact: “the circumference of a circle lies between the perimeters of the inscribed and circumscribed regular polygons of *n* sides, and as *n* increases, the deviation of the circumference from the two perimeters becomes smaller" (p. 202). This process is known as the method of exhaustion and would be beneficial for students to understand because this process was used by many mathematicians after Archimedes to estimate $\boldsymbol{\pi}$ until the aid of calculus and computers was utilized. This method involves fundamental concepts of geometry, such as perimeters and area of polygons. Archimedes' method of exhaustion can also be used as an introduction to limits or integration in a calculus class.

Historians study early civilizations' approximations of $\boldsymbol{\pi}$ because the level of accuracy represents the mathematical skills of the civilizations at that time, and the ancient Chinese were very advanced compared to other societies (Burton, 2007, p. 204). There were many ancient Chinese mathematicians who developed close approximations to $\boldsymbol{\pi}$. Around the year 23, Liu Hsin used 3.1547, and in the years (78-139), Chang Heng used $\sqrt{10}$. According to Burton (2007), "By taking the ratio of the perimeter of a regular inscribed polygon to the diameter of a circle enclosing the polygon, third century mathematicians obtained more accurate approximations" (p. 204). For example, Liu Hui used a 3072-sided polygon to find 3.14159 as his approximation for $\boldsymbol{\pi}$, and hundreds of years later, Ludolph van Ceulen calculated $\boldsymbol{\pi}$ to 35 decimals using a polygon of 2^{62} sides. Hereafter, mathematicians used calculus and then computers to evaluate $\boldsymbol{\pi}$; many connections can be seen between geometry and other courses. In 1593, Vieta used trigonometry to discover an infinite-product expansion for $\boldsymbol{\pi}$ in terms of square roots, which could be examined in a calculus class. Another discovery that could be investigated in a calculus course is how John Machin computed $\boldsymbol{\pi}$ to 100 decimal places using the identity $\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$ (Burton, 2007, p. 409). In 1949, John von Neumann computed $\boldsymbol{\pi}$ to 2037 digits using a computer that took 70 hours, and new algorithms have been developed that allows $\boldsymbol{\pi}$ to be carried out to millions of digits quickly. Many students memorize an estimate of $\boldsymbol{\pi}$, but the origins of $\boldsymbol{\pi}$ and an appreciation for the many attempts to approximate its value are lost (Howard, 2009, p. 334).

The Pythagorean Theorem is one of the most widely used theorems in all of mathematics, and high school students are expected to derive, extend, and use the Pythagorean Theorem (TEKS, 2009, p. 9). Many students believe Pythagoras discovered the theorem because it is named after him, but tablets from ancient civilizations, such as the Old Babylonians, show that the theorem was used and that the Babylonians had lists of Pythagorean triples. The theorem was also used in ancient Indian and Chinese civilizations. "It has been suggested, as justification for calling it the Theorem of Pythagoras, that the Pythagoreans first provided a demonstration; but this conjecture cannot be verified" (Boyer, 1991, p. 49). The Babylonians and Egyptians did mathematics to solve society's problems while the Greeks began to study mathematics purely. Pythagoras and his followers believed the dominance of math was the true source of knowledge, and his school was more like a secret society where some of their knowledge was only passed down to certain individuals (Mankiewicz, 2000, p. 24). In their secret society, many unusual customs were held in the name of number. The Pythagoreans followed a strict moral code and were forced to be vegetarians.

As previously stated, students need to be able to derive the Pythagorean Theorem. Throughout history, there have been many proofs of the Pythagorean Theorem. Euclid's proof, found in Book I of his *Elements*, demonstrates the Greeks' remarkable method of proof. According to Mankiewicz (2000), "It [Euclid's proof] is a very general geometric proof employing a sequence of constructions which transform the two smaller squares into two rectangles, which fit together to form the larger square" (p. 26). In "Not Just another Theorem: A Cultural and Historical Event," McDaniel discusses how mathematical proofs should be presented as museum pieces for students to verify their truths and how he used this method in his classroom for the Pythagorean Theorem. McDaniel (2003) claims the Pythagorean Theorem, which is approximately 2500 years old, and its proof should be presented in a similar atmosphere as bringing in the original Rosetta Stone into the classroom (p. 282). The teacher needs to convey the proof in an unforgettable manner to show its importance; this can be done by changing the location of the classroom to a more scholarly atmosphere, bringing in old mathematics books, maps, time lines, design tools, and anything involving the time period including wars, music, literature, art, and food (McDaniel, 2003, p. 282). Hopefully, students will begin to see the significance of this widely used theorem and view the theorem as a human creation. There are different ways to approach proving the theorem, but McDaniel (2003) suggests using cards with theorems, rules, and definitions written on them to build the proof with the class (p. 283). Teachers should make sure to create attractive presentations, cater to each of the class's personalities, and even insist students to dress up nicely or as the Pythagoreans. Dressing up nicely would be out of respect for the theorem because according to McDaniel (2003), "Establishing the absolute validity of an idea does not happen often in disciplines outside mathematics" (p. 283). Teachers should also demonstrate other proofs to compare methods, and have questions available to begin discussion. Doing this kind of activity shows the theorem's importance and value throughout history.

It was necessary for ancient civilizations to develop formulas for lengths, areas, and volumes of various shapes. The Egyptians needed formulas for surveying farmland and constructing the pyramids. In Mesopotamia, ancient Babylonians constructed clay tablets for trade and labor records, and formulas were necessary for their geometric abilities in architecture, agriculture, and astronomy. Ancient Chinese people focused on measuring fields and constructing buildings and walls, and in India, religious altars of various areas were built. "As for geometry, almost all of the early inhabitants of North, Central, and South America used geometric shapes for the construction of their houses and buildings and in textile and pottery design" (Beery, Dolezal, Sauk, Shuey, 2004, p. 25).

Having students investigate ancient mathematical formulas shows early civilizations' purposes for them. According to Howard (2009), "Ancient Egyptian mathematics lends itself readily to such overlapping of history and high school mathematics" (p. 332). Howard examines problems, such as approximating the area of a circle and π and the volume of a truncated pyramid, from ancient Egypt given in the Rhind Papyrus and the Moscow Papyrus. Students can do problems from the Moscow Papyrus that involve finding the volume of a frustum of a pyramid and problems from Babylonian clay tablets that also require finding the volume of a pyramid. Students are expected to apply appropriate techniques, tools, and formulas to determine measurements (NCTM, 2000, p. 320).

Studying ancient civilizations' shows the many needs they had for geometry including land surveying. Pi was required to take areas and volumes of certain figures, and the Pythagorean Theorem was used in measuring land and finding missing lengths. Studying the history of π and the Pythagorean Theorem reveals their true meanings. A sample lesson plan over the Pythagorean Theorem is given (see Appendix B: Geometry Lesson Plan). This lesson plan discusses the history of the Pythagorean Theorem and an introduction to proving the theorem. The researcher conducted a study using the geometry lesson plan to determine the effect of incorporating the history of the Pythagorean Theorem and is discussed at length later in the paper.

Example for Pre-Calculus

Trigonometry is a main component of a high school pre-calculus class. Students are required to examine some topics such as the graphs of trigonometric functions, the trigonometric functions' inverses, trigonometric identities, the law of sines, and the law of cosines. Students also learn about conic sections, logarithmic functions, and exponential functions in a pre-calculus class. The researcher will explore the history of these topics to present their purposes and give some activities that integrate the history to enhance student understanding.

Many ancient societies relied on shadow observations for agricultural and religious purposes (Swetz, 1995b, p. 57). For example, shadow lengths established solstices which gave civilizations knowledge to improve their planting seasons, and Egyptian and Hindu priests scheduled religious rituals according to the sun's position based on shadow lengths (Swetz, 1995b, p. 57). Ancient Egyptians had to use basic trigonometric concepts in land surveying and the building of the pyramids. Problem 56 of the Rhind Papyrus contains elements of trigonometry, and since the construction of the pyramids involved keeping a uniform slope for the faces, the Egyptians introduced a notion equivalent to the cotangent of an angle (Boyer, 1991, p. 18). Thales of Miletus (625-547 B.C.) is credited to the process of "shadow reckoning" to find the height of the pyramids in Egypt using a pole, known as a gnomon. According to Swetz (1995b), "With the simplest of instruments, a vertical staff of known length, and the natural phenomenon of sunlight casting shadows, Thales was able to utilize the principle of proportionality between similar right triangles to determine the heights of the pyramid" (p. 57). There are also problems in the Egyptian *Rhind Papyrus* that involve finding the inclination of a pyramid. Tangent and cotangent developed in calculating heights of objects by using the shadow lengths. A possible activity is for students to make a gnomon, record its height and shadow's height, and make a table of pole heights versus shadow heights (Crossfield, Shepherd, Stein, Williams, 2004, p. 32). This activity supports the geometry standard that students should recognize "right triangle trigonometry is useful in solving a range of practical problems" (NCTM, 2000, p. 313). Swetz (1995b) states that shadow ratios eventually led to trigonometric functions and probably gave a basis for the Pythagorean Theorem's discovery, and these facts need to be recognized in the teaching of trigonometry (p. 67).

The main driving force of the development of trigonometry was ancient civilizations' desires to understand astronomy, which depends on mathematics. They studied the movement of the stars and planets to understand the world around them. The Babylonians created tablets and prepared solar and lunar tables, which could be used to predict the appearance of the crescent moon (Mankiewicz, 2000, p. 18). According to Berlinghoff and Gouvêa (2002), the Greeks needed to be able to locate planets in the sky so they borrowed from Babylonian mathematics and began using numbers to measure angles (p. 21). This is where the beginning of trigonometry can be seen. Some significant Greek astronomers who helped develop the foundation of trigonometry are Hipparchus and Ptolemy. Early Indian mathematics was inspired by Greek mathematics, and Indian mathematics developed as a product of astronomy and what has come down to us appeared as chapters in astronomical works (Burton, 2007, p. 227).

According to the NCTM (2000), instructional programs should allow students to recognize and apply mathematics in contexts outside of mathematics (p. 354). Students can do an activity involving Aristarchus' method of finding the distance from the Sun to the Earth, which demonstrates an introduction to the development of trigonometry and its connection to areas outside of math. Aristarchus asserted that the sun was between eighteen and twenty times as far from the earth as is the moon, which was much less than the modern value of less than 400, but his method was extremely reliable (Boyer, 1991, p. 160). Aristarchus also used angles, similarity of triangles, and ratios to determine the proportions of the radius of the earth to the radius of the moon and of the radius of the sun to the radius of the earth. Students can also explore Eratosthenes' method for the measurement of the earth by using lengths of shadow of a pole and angles. This was one of the most successful estimates made in ancient times.

Hipparchus is known as the "father of trigonometry," and his basis of trigonometry was the division of the circle into 360 degrees (Mankiewicz, 2000, p. 18). Trigonometric tables were generated for computations in astronomy, and Hipparchus was the first person to create such a table of ratios (Adamek, Penkalski & Valentine, 2005, p. 3). These tables were written in twelve of his books, which none survived, and found the length of a subtended chord (a sine) given a circular arc (Crossfield et al., 2004). According to Mankiewicz (2000), "A chord is essentially double the sine of half an angle" (p. 20). Other great works of Hipparchus include approximating Earth's radius and the sun and moon's diameters. These activities can be used in the classroom, and they support the problem solving standard, which states that students should "solve problems that arise in mathematics and in other contexts" (NCTM, 2000, p. 334). These activities demonstrate the purpose of the development of trigonometry.

Another influential figure in the development of trigonometry was Ptolemy (c. 100 A.D. – 170). Ptolemy was an Egyptian mathematician, astronomer, and geographer, and his first major work was the *Almagest*, which provides his astronomical observations. His contributions to trigonometry are important because his table of lengths of chords in a circle is the earliest surviving table of a trigonometric function ("Ptolemy," 2010, p. 3). Several activities involving the work of Ptolemy can be used in a pre-calculus class. Ptolemy's table of chords is said to involve sine. This can be shown by applying the sine ratio to a central angle and its chord (Crossfield et al., 2004). Also, Ptolemy's Theorem states that if ABCD is a quadrilateral inscribed in a circle, then the product of the diagonals is equal to the sum of the products of the two pairs of opposite sides: AC · BD = AB · CD + BC · AD. This theorem leads to the trigonometric formula $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$. Using substitutions, students then can derive the half angle formulas for sine and cosine. Using Ptolemy's Theorem to derive trigonometric formulas promotes the reasoning and proof standard that students should "recognize reasoning and proof as fundamental aspects of mathematics" (NCTM, 2000, p. 342).

Hindu and Arabic mathematicians also made great contributions to the development of trigonometry, which can be studied by high school students. Hindu mathematicians replaced the Greek tables of chords with an introduction of an equivalent of the sine function, which can be found in the *Siddhāntas* and the *Aryabhatiya* (Boyer, 1991, p. 215). Brahmagupta was an influential Indian mathematician and found that $2r = \frac{a}{\sin A}$ which also equaled $\frac{b}{\sin B}$ and $\frac{c}{\sin C}$ (Crossfield et al., 2004). He also developed his own theorem to find the area of a cyclic quadrilateral using its sides, and students can prove his theorem using the Law of Sines. Arabic trigonometry was also built on the sine function. According to Boyer (1991), "The law of sines had been known to Ptolemy in essence and is implied in the work of Brahmagupta, but it frequently is attributed to Abu'l-Wefa because of his clear-cut formulation of the law for spherical triangles" (p. 238). Shadow lengths were also used in India and Arabia. There was a general theory about shadow lengths, as related to a unit of length for varying solar altitudes (Boyer, 1991, p. 238). The common trigonometric functions used today (tangent, cotangent, secant, and cosecant) were used in "shadow reckoning." For example, the horizontal shadow, for a vertical gnomon of given length, was what we call the cotangent of the angle of elevation of the sun, and the "reverse shadow," the shadow cast on a vertical wall by a gnomon projecting horizontally from the wall, was the tangent of the solar elevation (Boyer, 1991, p. 238). Also, the equivalents of cosecant and secant were found. Boyer (1991) states,

> The "hypotenuse of the shadow" – that is, the distance from the tip of the gnomon to the tip of the shadow" – was the equivalent of the cosecant function; and the "hypotenuse of the reverse shadow" played the role of our secant. (p. 238)

If the activity of "shadow reckoning," previously discussed in this section, is used in the classroom, this process of finding the tangent, cotangent, secant, and cosecant, just as ancient Indians and Arabians found these values, should be accomplished. This activity supports the curriculum principle because it deepens students' admiration of mathematics as a discipline and human creation by providing the beginning and purpose of trigonometry (NCTM, 2000, p. 15).

François Viète is another important figure in the history of trigonometry. The trigonometry of Viète was distinguished by an amplified emphasis on generality, and he can be called father of a generalized analytic approach to trigonometry (Boyer, 1991, p. 307). He thought of trigonometry as a separate branch of mathematics and had tables of all six trigonometric functions for angles. Viète also developed a statement equivalent to our law of tangents (Boyer, 1991, p. 308). In addition, he made connections with trigonometry to other areas of mathematics. For example, Viète continued to use Ptolemy's formulas recursively to derive a formula for $\sin nx$ and $\cos nx$, which demonstrates a link between trigonometry and number theory. Students should "understand how mathematical ideas interconnect and build on one another to produce a coherent whole" (NCTM, 2000, p. 354), and studying Viète would assist them with their understanding.

Conic sections, which include circles, ellipses, parabolas, and hyperbolas, are another part of a pre-calculus class. The Greek astronomer Apollonius is best known for his work *Conic Sections*, in which he discusses the series of curves formed when a plane surface intersects a cone (Smith, 1996, p. 31). According to Smith (1996):

> Different figures are formed by this intersection, depending on *where* the plane intersects the cone: parallel to the base (a circle), oblique to the base (an ellipse), such that it intersects the base (a parabola), and parallel to the altitude of the cone (a hyperbola). (p. 31)

Studying the history of conic sections shows their purposes. For example, the Greeks used them in geometric constructions. Also, Galileo Galilei found that the parabola illustrates earth's trajectiles, and Edmond Halley used data from comets to conclude that they represented a single comet orbiting the sun every 76 years in an elliptical path (Smith, 1996, p. 31). Students are expected to use conic sections to model motion, such as the motion of planets (TEKS, 2009, p. 11). Students can also investigate examples that conic sections describe in our world such as a sonic boom, satellite dishes, and reflecting telescopes.

Logarithmic functions are studied in a high school pre-calculus class. During the Renaissance, trigonometry had become its own branch, and working with difficult trigonometric tables led to the invention of logarithms by John Napier (Burton, 2007, p. 340). Napier's logarithms differ from the ones used today, but according to Katz (1995), Napier's original method can be used in today's classrooms to introduce natural logarithms to pre-calculus students (p. 49). In addition, the use of sequences is used to present Napier's original methods. Logarithms are very applicable today in finance, engineering, astronomy, and biology. An interesting point for students could be that there was some controversy about who invented logarithms. Jobst Bürgi claimed to be the inventor, but Napier is considered the true inventor because he made his announcement of his discovery six years before Bürgi published a table of logarithms (Smith, 1996, p. 77).

Trigonometry is a major focus in pre-calculus courses, and exploring its history shows the purpose of trigonometry, which may not always be so clear to students. Other topics such as conic sections and logarithmic functions are also discussed in pre-calculus classes, and its history demonstrates its use in everyday life. A sample lesson plan about using shadow lengths to introduce the tangent function is given to show how trigonometric functions were developed (Appendix C: Pre-Calculus Lesson Plan).

Example for Calculus

Calculus has an interesting history and many uses in past society and today's society; both should be studied in the classroom to show its development and purpose. Students should be given problems that apply to the natural sciences, and historical developments should be discussed with students to view mathematics as a discipline that has steadily grown over many centuries (Helfgott, 1995, p. 136).

Teachers can have students do several activities to learn more about the history of calculus and its importance. Having students research the mathematicians who are precursors to Newton and Leibniz will help them learn about the foundation of calculus. Some of these include John Wallis, Isaac Barrow, Christiaan Huygens, and René Descartes (Smith, 1996, p. 93). This research will introduce students to calculus and ignite some interest which will lead to making other mathematical connections and understanding how calculus is a human creation, supporting the learning principle. Putting students into groups and researching in the library and on the internet is a great way to introduce the calculus course. Some questions for them to research include: what is calculus, why calculus was developed, who developed calculus, the development of notation, the calculus Newton-Leibniz controversy, and applications of calculus. These questions support the teaching principle that states that well-chosen tasks connect to real-world experiences and build understanding (2000, p. 19). The following is information that students would find.

Isaac Newton and Gottfried Leibniz are often credited with the invention of calculus, but there are many other historical figures that laid the groundwork for them to discover calculus. According to Cirillo (2007), mathematicians such as Archimedes, Kepler, Descartes, Fermat, Pascal, and Barrow had already developed Newton and Leibniz's basic elements of calculus. It would be beneficial for students to conduct some research on some predecessors of Newton and Leibniz to realize how extremely lengthy the road to reach the invention of modern calculus was. This research may help students if they are struggling with a certain lesson in calculus because they will see how it took thousands of years to develop calculus; therefore, it is okay if students do not understand it all in that one day.

Isaac Newton was born in the same year that Galileo died – 1642, and was raised by his grandmother. An uncle convinced Newton's mom to enroll him at Cambridge; at first he never intended on being a mathematician but instead studied chemistry (Boyer, 1991, p. 391). However, the works of influential historical mathematicians sparked interest in Newton. After developing a deep foundation in mathematics, Newton began to work on his own contributions to the field of study. Between 1665-1666, Newton's college was shut down because of the plague so he went home to think. During this time, he was very productive for he had discovered "1) the binomial theorem, 2) the calculus, 3) the law of gravitation, and 4) the nature of colors" (Boyer, 1991, p. 393). According to Cirillo (2007), Gottfried Leibniz was born into an educated family and had received a doctorate in law by the age of 20. In 1676, Leibniz had begun to develop differential calculus by studying infinite series (Boyer, 1991, p. 400). His notation for calculus is still used today.

In 1695, the controversy over who discovered calculus began. In 1695, Wallis told Newton that in Holland, they believed Leibniz was the inventor of calculus. However, by 1699, de Duillier had written in a paper that "Leibniz may have taken his ideas on the calculus from Newton" (Boyer, 1991, p. 414). Leibniz protested against the Royal Society's accusations of plagiarism. Eventually, the Royal Society declared Newton the first inventor of calculus (Boyer, 1991, p. 414). Today, they are both credited as the inventors of calculus. The researcher believes this controversy will interest students and spark some curiosity over the significance of calculus and why it would create such a debate.

Many benefits for the student will come from including the history of math in a calculus class. Studying the history of math will show students why calculus was developed. "When astronomers in Europe provided evidence that the earth is not the center of the universe and that the orbits of planets are not circular, new mathematics was needed to explain natural events and the motion of heavenly bodies" (Smith, 1996, p. 93). It is hopeful that students will find this example interesting and realize that math has purpose; just because they may not use it every day does not mean it is useless. In fact, many professions apply all aspects of math to various fields such as physics and engineering.

Calculus can be a difficult subject for students, but its history can provide clarification. According to Rickey (1995), historical notes in calculus can captivate students, and anecdotes and biographies make problems more interesting (p. 133). Also, history stimulates students to want to do more mathematics and can be essential to understanding (Rickey, 1995, p. 133). A sample calculus lesson plan that inspects the history of the product rule was created (see Appendix D: Calculus Lesson Plan).

Method of the Study

A study was conducted in conjunction with the writing of this paper. The researcher's goal was to determine whether or not students benefitted from learning about the history of the Pythagorean Theorem and if it helped them understand the theorem more thoroughly. This study was done in a mathematics education class consisting of future elementary teachers. An attitudinal survey was given (see Appendix E: Pre-Lesson Survey). The survey consisted of questions that asked about students' current knowledge of the Pythagorean Theorem, if they knew where it came from and their opinions on the effects of learning about the history of math. The possible answers ranged from strongly disagree to strongly agree, and this survey was given the day before the lesson. The following day, the researcher presented a lesson on the Pythagorean Theorem incorporating its history into the lesson. First, a PowerPoint (see Appendix F: Pythagorean Theorem PowerPoint) was presented that explained the theorem and how to use it, including two example problems. The researcher also discussed ancient civilizations that developed the Pythagorean Theorem before Pythagoras such as the Babylonians, Chinese, and Egyptians; Pythagoras and his school; and the theorem's many proofs. After the PowerPoint, a Pythagorean Theorem dissection puzzle was handed out for students to solve. The puzzle entailed fitting both the square of side a and the square of side b of a right triangle into the square of the hypotenuse c (see Appendix G: Pythagorean Theorem Puzzles). This gave the students an introduction to proving the theorem. Once students determined how to solve the puzzles, the researcher presented Bhaskara's proof of the Pythagorean Theorem (see Appendix H: Pythagorean Theorem Proofs) and explained how he proved it. Next, President Garfield's Pythagorean Theorem proof was demonstrated (see Appendix H: Pythagorean Theorem Proofs), and some historical background information on

President Garfield was given. After the lesson was finished, the same attitudinal survey was given as a post-lesson survey to the students. The post-lesson survey was given to answer to determine their changes in attitude (see Appendix I: Post-Lesson Survey). Lastly, an assessment of four questions was given to the students (see Appendix J: Pythagorean Theorem Assessment). It included questions about the history of the Pythagorean Theorem, questions about finding the missing lengths of right triangle, and a figure given to be used to develop a proof of the Pythagorean Theorem.

Treatment of the Data

For the pre-lesson survey and the post-lesson survey, the researcher examined the surveys and tallied how many students answered strongly agree, agree, neutral, disagree, and strongly disagree for each of the five questions. Then, a percentage was calculated for each rating for each question. The researcher also observed each student's pre-lesson survey and post-lesson survey and recorded the ratings for each survey to note the positive, negative, or neutral change in ratings. The researcher then evaluated the lesson assessments and determined how many problems each student answered correctly. Then, the percentage of the correct answers per question was calculated.

Results of the Study

Overall, the researcher believes the results of the study support the theme throughout the thesis paper that the history of mathematics improves student understanding.

Table 1: Pre-Lesson Survey Results

Question	Strongly Agree	Agree	Neutral	Disagree	Strongly Disagree
1	5.3%	36.8%	26.3%	15.8%	15.8%
2	0%	15.8%	31.6%	31.6%	21.1%
3	5.3%	57.9%	15.8%	5.3%	15.8%
4	26.3%	52.6%	10.5%	0%	10.5%
5	31.6%	57.9%	10.5%	0%	0%

Examining the change in ratings of the pre-lesson survey and the post-lesson survey of question one, which stated, "My knowledge of the Pythagorean Theorem is strong," eleven ratings increased, seven remained the same, and one decreased. The seven ratings that remained the same were all either strongly agree, agree, or neutral, which the researcher considers positive because no answer of disagree or strongly disagree on the pre-lesson survey was answered the same on the post-lesson survey. For question two, which stated, “I understand where the Pythagorean Theorem comes from,” seventeen ratings increased and two remained the same. The researcher considers students to have truly learned the history of the Pythagorean Theorem since seventeen of the nineteen students increased their ratings. The third question, which stated, “I know how to use the Pythagorean Theorem,” resulted in eleven ratings increasing, six remaining the same, and two decreasing. Of the six ratings that remained the same, five were answered agree on both surveys, and one was answered neutral on both survey. It is significant that the majority of the students felt they learned how to use the theorem because it is imperative for students to know how to apply the theorem instead of just knowing the formula.

Table 2: Post-Lesson Survey Results

Question	**Strongly Agree**	**Agree**	**Neutral**	**Disagree**	**Strongly Disagree**
1	31.6%	26.3%	26.3%	15.8%	0%
2	36.8%	47.4%	15.8%	0%	0%
3	36.8%	36.8%	21.1%	5.3%	0%
4	36.8%	52.6%	10.5%	0%	0%
5	31.6%	47.4%	10.5%	10.5%	0%

For question four, which stated, “I think learning about the history of math shows math’s purpose and portrays math as a human creation,” six ratings increased, nine ratings remained the same, and four decreased. The nine ratings that remained the same were the answers strongly agree, agree, and neutral, which once again, the researcher considers to be positive.

Of the four ratings that decreased, one rating went from agree to neutral, and the other three ratings decreased from strongly agree to agree. Therefore, the researcher does not necessarily view the four decreased ratings as a negative result. For question five of the survey, which stated, "Learning the history of Pythagorean Theorem helps me understand it more thoroughly," five ratings increased, seven ratings remained the same, and seven ratings decreased. The seven ratings that remained the same were either strongly agree, agree, or neutral, which the researcher does not consider to be a negative result. It may seem that the seven decreased ratings are negative results, but of those seven decreased ratings, four ratings decreased from strongly agree to agree, one rating decreased from strongly agree to neutral, and one rating decreased from neutral to disagree. The researcher believes only the two ratings that decreased from strongly agree to neutral and from neutral to disagree are negative results.

Table 3: Change in Ratings of Pre-Lesson Survey and Post-Lesson Survey

Question	Increased 4	Increased 3	Increased 2	Increased 1	Same	Decreased 1	Decreased 2
1	0	1	2	8	7	1	0
2	1	4	6	6	2	0	0
3	0	2	2	7	6	2	0
4	1	1	1	3	9	4	0
5	0	0	0	5	7	5	2

The researcher considers the students to have done well on the lesson assessment. Question one, which involved finding the missing length of a side of a right triangle using the Pythagorean Theorem, was answered correctly by 100% of the students. Question two, which asked who can be given priority to discovering the theorem, was also answered correctly by

100% of the students. Question three, which asked students to prove the theorem using a given figure, was answered correctly by 65% of the students. The researcher believes this percentage is lower because proofs take practice to excel at, and students were given the quiz directly after the lesson. Question four, which asked students what the motto the Pythagoreans lived by was, was answered correctly by 85% of the students.

Table 4: Quiz Results

Question	**% answered correctly**
1	100%
2	100%
3	65%
4	85%

Suggestions for Further Research

The researcher would like to see the following questions explored for future research.

- How beneficial is the history of mathematics incorporated into the postsecondary mathematics curriculum?
- How beneficial is the history of mathematics incorporated into elementary mathematics classes?
- Do students truly enjoy learning the history of mathematics?

Bibliography

Activities: Pythagorean Dissection Puzzles. (1993). *Mathematics Teacher* , 302-308, 313-314.

Adamek, T., Penkalski, K., & Valentine, G. (2005, May 11). *History of Trigonometry.* Retrieved July 5, 2010, from Rutgers Web site: http://www.math.rutgers.edu/~mjraman/History_Of_Trig.pdf

Algebra II Clarifying Activities. (n.d.). Retrieved June 15, 2010, from Mathematics TEKS Tool Kit: http://www.utdanacenter.org/mathtoolkit/

Avital, S. (1995). History of Mathematics Can Help Improve Instruction and Learning. *Learn from the Masters* , 3-12.

Bashmakova, I., & Smirnova, G. (2000). *The Beginnings and Evolution of Algebra.* United States of America: The Mathematical Association of America.

Beery, J., Dolezal, C., Sauk, A., & Shuey, L. (2004). Lengths, Areas, Volumes. In V. J. Katz, & K. D. Michalowicz, *Historical Modules for the Teaching and Learning of Mathematics* (pp. 6-32). The Mathematical Association of America.

Berlinghoff, W. P., & Gouvea, F. Q. (2002). *Math through the Ages.* Farmington: Oxton House Publishers.

Boyer, C. B. (1991). *A History of Mathematics.* New York: John Wiley & Sons, Inc.

Burton, D. (2007). *The History of Mathematics.* New York: McGraw-Hill.

Cirillo, M. (2007). Humanizing Calculus. *Mathematics Teacher* , 23-27.

College Board AP. (2010). *Calculus Course Description.* Retrieved May 3, 2010, from College Board AP Web site: http://apcentral.collegeboard.com/apc/public/repository/ap-calculus-course-description.pdf

Crossfield, D., Shepherd, C., Stein, R., & Williams, G. (2004). Trigonometry. In V. J. Katz, & K. D. Michalowicz, *Historical Modules for the Teaching and Learning of Mathematics* (pp. 8-11, 28-33, 37-47). The Mathematical Association of America.

Furinghetti, F. (1997). History of Mathematics, Mathematics Education, School Practice: Case Studies in Linking Different Domains. *For the Learning of Mathematics* , 55-61.

Helfgott, M. (1995). Improved Teaching of the Calculus Through the Use of Historical Materials. *Learn from the Masters* , 135-144.

Howard, C. A. (2009). Mathematics Problems from Ancient Egyptian Papyri. *Mathematics Teacher* , 332-339.

Jardine, R. (1997). Active Learning Mathematics History. *Primus* , 115-121.

Katz, V. J. (1995). Napier's Logarithms Adapted for Today's Classroom. *Learn from the Masters* , 49-55.

Katz, V. (2007). Stages in the History of Algebra with Implications for Teaching. *Educational Studes in Mathematics* , 185-201.

Kelley, L. (2000). A Mathematical History Tour. *The Mathematics Teacher* , 14-17.

Lewinter, M., & Widulski, W. (2002). *The Saga Mathematics.* Upper Saddle River: Prentice Hall.

Livio, M. (2002). *The Golden Ratio.* New York: Broadway Books.

Mankiewicz, R. (2000). *The Story of Mathematics.* Princeton: Princeton University Press.

Marshall, G. L., & Rich, B. S. (2000). The Role of History in a Mathematics Class. *The Mathematics Teacher* , 704-706.

McBride, C. C., & Rollins, J. H. (1977). The Effects of History of Mathematics on Attitudes toward Mathematics of College Algebra Students. *Journal for Research in Mathematics Education* , 57-61.

McDaniel, M. (2003). Not Just Another Theorem: A Cultural and Historical Event. *Mathematics Teacher* , 282-284.

Principles and Standards for School Mathematics. (2000). Reston: The National Council of Teachers of Mathematics, Inc.

Ptolemy. (2010). Retrieved July 6, 2010, from Encylopedia Britannica: http://search.eb.com/eb/article-253371

Rickey, V. F. (1995). My Favorite Ways of Using History in Teaching Calculus. *Learn from the Masters* , 122-134.

Smith, S. (1996). *Agnesi to Zeno: Over 100 Vignettes from the History of Math.* Emeryville: Key Curriculum Press.

Swetz, F. J. (1995). Trigonometry Comes Out of the Shadows. *Learn from the Masters* , 56-71.

Swetz, F. J. (1995). Using Problems form the History of Mathematics in Classroom Instruction. *Learn from the Masters* , 24-38.

Texas Essential Knowledge and Skills for Mathematics. (2009, February 23). Retrieved June 1, 2010, from Texas Education Agency Web site: http://ritter.tea.state.tx.us/rules/tac/chapter111/ch111c.html

Wilson, P. C., & Chauvot, J. B. (2000). Who? How? What? A Strategy for Using History to Teach Mathematics. *The Mathematics Teacher* , 642-645.

Appendix A: Algebra Lesson

Lesson Plan: Subject: Algebra

Teacher: Lacy Gaines **Grade Level: 9th** **Date: July 28, 2010**

Overall Goal of the lesson: To utilize al-Khwarizmi's completing the square methods

Cognitive Objective (Based on TEKS): The student will be able to: use the method of completing the square to solve quadratic equations. Students will perceive connections between algebra and geometry and use the tools of one to help solve problems in the other (Section 111.33a4).

Differentiated Learning: (Based on Bloom's Taxonomy)

Linguistic (listen, speak, read, write): The student will: listen to the teacher discuss some background information on al-Khwarizmi and his contributions to algebra, speak with a partner to solve problems, and write to work out problems and solutions.

Affective (Social/Emotional – cooperative learning): The student will participate in: partner work to solve quadratic equations using the completing the square method.

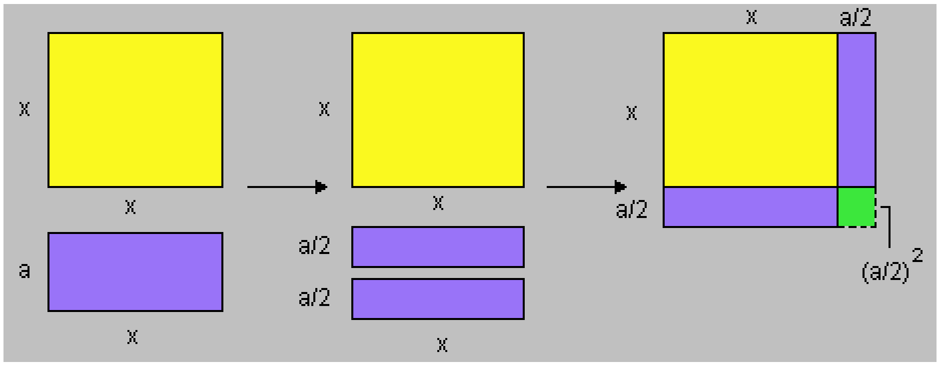
Psychomotor (kinesthetic learning): The student will participate in: an activity that uses algeblocks to provide a visual justification for completing the square.

Materials/Equipment Needed: algeblocks and algeblock mats

Anticipatory Set: The teacher should have students play around with the algeblocks for a few minutes, especially if it is their first time to use them. Have them factor simple quadratic equations using the algeblocks. Lastly, give them a quadratic equation to solve that cannot be solved by factoring. This is where the need for completing the square develops.

Instructional Input: The teacher will discuss al-Khwarizmi and his contributions to algebra, specifically his method of completing the square. The ancient Babylonians' method of completing the square should also be discussed. Have students model the quadratic equation with algeblocks and use the blocks to form a square. Students should then complete the square by adding the appropriate amount of units. Then translate the process symbolically to find the solutions of the quadratic equation.

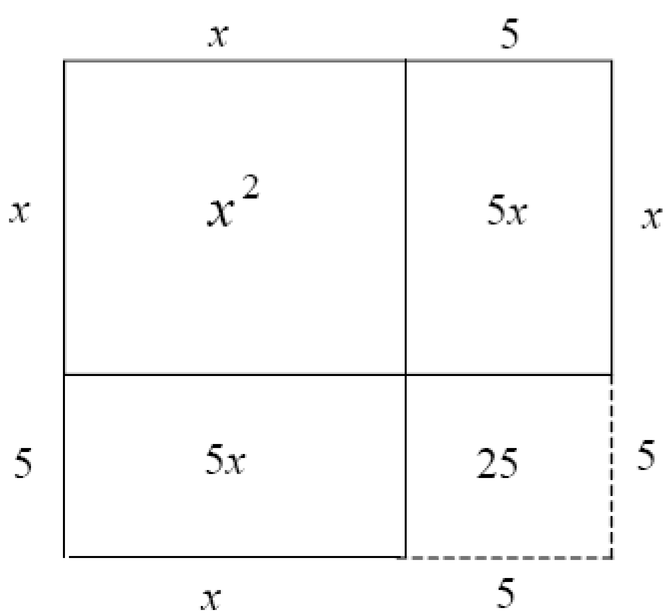
For a simple situation, students model the process of completing the square. For example, to complete the square for the expression $x^2 + ax$, students create a model like the one below to show that a term of $(a/2)^2$ must be added.



(http://www.utdanacenter.org/mathtoolkit/instruction/activities/alg2.php)

Modeling: The teacher should do an example, such as $x^2 + 10x = 39$, by completing the square using the algeblocks and solving it symbolically.

The equation $x^2 + 10x = 39$ was one that al-Khwarizmi wanted to solve and used the technique of completing the square:



Now if we compute the area piece by piece, the completed square has area $x^2 + 5x + 5x + 25 = (x^2 + 10x) + 25 = 39 + 25 = 64$. So $(x + 5)^2 = 64$, and $x = 3$.

Check for Understanding:

- What does each shape of algebra block represent?
- How can we represent a set of algebra blocks?
- How can we use the algebra blocks to represent a polynomial expression?
- How can we use the algebra blocks to represent a product of two polynomial factors?

- When a quadratic equation cannot be represented as a rectangle, what can we add to make it a perfect square?

Guided Practice: Students should be given a couple of quadratic equations to be solved using their algeblocks. Some can be specifically from al-Khwarizmi or the Babylonians.

For example, here is a problem from 2000 B.C. from the ancient Babylonians:

I have added the area and 2/3 the side of my square and it is 35/60. What is the side of my square?

Closure: The teacher should answer any lingering questions. Then the teacher should review the history of al-Khwarizmi and his contributions to algebra, and the ancient Babylonians' method. The many methods of solving quadratic equations and its long history should be also be emphasized. In addition, the teacher should discuss the connections between algebra and geometry.

Evaluation Strategy: A quiz should be given the following day consisting of quadratic equations to be solved using the completing the square method. Algeblocks can be used if needed. Also, questions about al-Khwarizmi and his contributions to algebra should be asked. The homework assignment will also be used to evaluate the students' knowledge of solving quadratic equations by completing the square.

Independent Practice: A homework assignment should be given that contains quadratic equations to be solved by completing the square and other geometric algebra problems solved by al-Khwarizmi.

Resource: http://www.utdanacenter.org/mathtoolkit/instruction/activities/alg2.php; Historical Modules, Length Area Volume, al-Khwarizmi's Completing the Square, Janet Beery, Connie Dozal, Angi Sauk, Linda Shuey

Appendix B: Geometry Lesson

Lesson Plan: Subject: Geometry

Teacher: Lacy Gaines **Grade Level: 10th** **Date: July 28, 2010**

Overall Goal of the lesson: To learn about the history of the Pythagorean Theorem, work Pythagorean puzzles to introduce students to proofs, and learn some historical Pythagorean Theorem proofs.

Cognitive Objective (Based on TEKS): The student will be able to: derive, extend, and use the Pythagorean Theorem (Section 111.34 8c).

Differentiated Learning:

Linguistic (listen, speak, read, write): The student will: listen to the teacher explain the development of the Pythagorean Theorem, speak to classmates to solve Pythagorean puzzles that visually prove the theorem and do proofs, read information on PowerPoint discussing the Pythagorean Theorem, and write down notes and proofs.

Affective (Social/Emotional – cooperative learning): The student will participate in: paired activity to solve puzzle and develop proofs of the Pythagorean Theorem.

Psychomotor (kinesthetic learning): The student will participate in: paired activity to solve Pythagorean puzzle by moving shapes around to prove the square of side a and the square of side b is equal to the square of side c.

Materials/Equipment Needed: projector, computer with PowerPoint, scissors, tape

Anticipatory Set: The Pythagorean puzzles can be used as an anticipatory set. Have students work individually or in pairs and follow the directions given on the worksheet. Students will cut the squares of side a and side b into several shapes to fit into the square of c. This is a good visual justification of the Pythagorean Theorem.

Instructional Input: The teacher will present the PowerPoint over the history of the Pythagorean Theorem, discussing its use in several ancient civilizations, such as the Babylonians and the Greeks.

Modeling: The teacher will demonstrate how to prove the Pythagorean Theorem using the figure used by Bhaskara and the ancient Chinese, involving areas of squares and triangles with lengths a, b, and c. The area of the square will be found two different ways, and then those two areas will be equated to obtain $a^2 + b^2 = c^2$.

Check for Understanding:

- What is the area of the square written in two different ways? c^2 and $(b – a)^2 + 4(\frac{1}{2}ab)$
- Are these equal? If so, set them equal to one another, and simplify.

Guided Practice: Students will then be asked to prove the Pythagorean Theorem using President Garfield's figure – a trapezoid formed by three right triangles – by finding the area of the trapezoid in two different ways and then equating them.

Closure: The teacher should review the history of the Pythagorean Theorem and emphasize that Pythagoras did not discover it, but his school did much to increase interest in problems entailing the theorem. The theorem's purpose in the ancient everyday life and its value throughout history should also be discussed.

Evaluation Strategy: A quiz will be given that asks students to prove the Pythagorean Theorem using a given figure, to solve for the missing lengths of a right triangle, and some questions involving the history of the theorem. Students will also be evaluated on their homework assignment.

Independent Practice: Homework problems will be given to students that asks for missing lengths of a right triangle, to prove whether or not a given triangle with given lengths is a right triangle using the converse of the Pythagorean Theorem, and to prove the Pythagorean Theorem using given figures. Students can also research to find other historical proofs.

Resource: Activities: Pythagorean Dissection Puzzles, *Mathematics Teacher*, April 1993; "Pythagoras and President Garfield" – *Agnesi to Zeno: Over 100 Vignettes from the History of Math* by Sanderson Smith.

Appendix C: Pre-Calculus Lesson

Lesson Plan: Subject: Pre-Calculus

Teacher: Lacy Gaines **Grade Level: 11th** **Date: June 28, 2010**

Overall Goal of the lesson: The overall goal of this lesson is for students to be introduced to the tangent function as the ratio of the length of a vertical pole to the length of a shadow created by that pole, from a light source.

Cognitive Objective (Based on TEKS): The student will be able to: use functions as well as symbolic reasoning to represent and connect ideas in geometry, probability, statistics, trigonometry, and calculus, model physical situations, and use functions and their properties, tools and technology, to model and solve meaningful problems. (Section 111.35 b 1, Section 111.35 c 3)

Differentiated Learning: (Based on Bloom's Taxonomy)

Linguistic (listen, speak, read, write): The student will: listen to the teacher give instructions and model the procedure, speak with partner to complete the activity, read instructions given, and write down data.

Affective (Social/Emotional – cooperative learning): The student will participate in: partner activity to develop tables of ratio like ancient civilizations did.

Psychomotor (kinesthetic learning): The student will participate in: an activity that involves setting up poles and measuring their shadows using a ruler or measure tape.

Materials/Equipment Needed: poles or sticks, rulers or measuring tape, graph paper

Anticipatory Set: Students can be given a few right triangles, and have the students solve certain angles or side lengths using the tangent function. This allows them to review the tangent function and help them connect to the following activity.

Instructional Input: The teacher should tell students the following instructions and print them out for them to refer to as they are doing the activity.

1. Set up poles of different lengths vertically touching the ground.
2. Measure the shadow the pole produces.
3. Create a table of pole heights vs. shadow lengths.
4. Graph the data.
5. Find the ratio of length of the pole to the shadow length, and write the fraction in reduced form.
6. Answer the given questions using the graph.

An example of a question could be, “How long a shadow does a six foot person cast?”

Modeling: The teacher can model the activity by using one of the poles and its shadow length to complete all of the required steps so students know how to perform the activity.

Check for Understanding:

- Do you notice any similarities between the ratios of the pole lengths and shadow lengths?
- What does this activity have to do with the tangent function?
- How can we use the tangent function?
- Why did ancient civilizations do this activity of shadow reckoning?

Guided Practice: As students are doing the shadow reckoning activity, the teacher should walk around to keep students on task and answer any questions students may have. When students begin working out the problem given to solve using their graphs, the teacher should be available to answer any questions or help them get started on the problem.

Closure: The teacher should review the purpose of the activity – to learn about the tangent function, construct a table like the ancient civilizations’ tables, and ancient civilizations’ use for shadow lengths in agriculture, religion, and astronomy. These ancient civilizations include the Babylonians, Chinese, Egyptians, and Greeks.

Evaluation Strategy: Students should be evaluated on whether or not they followed instructions properly and the accuracy of their graphs. They can also be evaluated on the given questions and a quiz given the next day.

Independent Practice: Students will be given a homework assignment consisting of problems from the history of shadow reckoning. These questions can come from the Rhind Papyrus, can be to find heights of flagpoles or buildings, can be to find the circumference of the earth using Eratosthenes’ method, or can be to develop trigonometric identities using geometry like Aryabhata.

For example:
Problem #58 from the Rhind Papyrus
If a square pyramid is $93\frac{1}{3}$ cubits high and the side of its base is 140 cubits long, what is its seqt?
(The seqt is equivalent to the cotangent ratio determined by an angle.)

Resource: Shadow Reckoning, Classroom Activity #1, Historical Modules, Don Crossfield, Charlyn Shepherd, Robert Stein, Grace Williams; Trigonometry Comes Out of the Shadows, Frank Swetz

Appendix D: Calculus Lesson

Lesson Plan Subject: Calculus

Teacher: Lacy Gaines **Grade Level: 12**

Overall Goal of the lesson: The overall goal of this lesson is to use the history of math to learn the product rule and the quotient rule, discovered by Gottfried Leibniz .

Cognitive Objective (Based on TEKS): The student will be able to: understand the meaning of the derivative in terms of a rate of change and local linear approximation and should be able to use derivatives to solve a variety of problems; students should develop an appreciation of calculus as a coherent body of knowledge and as a human accomplishment. (§111.54. Advanced Placement (AP) Calculus AB, Goals of Calculus AB by the College Board, Topic II Derivatives: Concept of the Derivative)

Differentiated Learning: (Based on Bloom's Taxonomy)

Linguistic (listen, speak, read, write): The student will: listen as the teacher demonstrates the application of the product rule and the quotient rule, discuss with other students as they research, read while researching, and write down notes.

Affective (Social/Emotional – cooperative learning): The student will participate in: group activity as each group researches to find the product rule and quotient rule.

Materials/Equipment Needed: Computers with Internet access, calculus books, pencil, paper, history of math books from library

Anticipatory Set: In 1675, Leibniz began to investigate the basic algorithms of calculus. He wondered if the derivative of a product of two functions is equal to the derivative of the first function multiplied by the derivative of the second function. If $F(x) = f(x)g(x)$, does $F'(x) = f'(x)g'(x)$? Also, he wondered if the derivative of a quotient of two functions is equal to the quotient of the derivative of the numerator and the derivative of the denominator. If $F(x) = \frac{f(x)}{g(x)}$, does $F'(x) = \frac{f'(x)}{g'(x)}$? Have students multiply $(3x^2 + 2)(x + 5)$ and find the derivative. Does this equal $(3x^2 + 2)'(x + 5)'$? Have students divide $\frac{3x^3+15x^2+2x+10}{x+5}$ and find the derivative. Does this equal $\frac{(3x^3+15x^2+2x+10)'}{(x+5)'}$?

Instructional Input: Once students realize that $F'(x) \neq f'(x)g'(x)$ and $F'(x) \neq \frac{f'(x)}{g'(x)}$, have students research through books or the Internet to find a process that allows us to find derivatives of products and quotients more efficiently than multiplying and dividing the functions out and then taking the derivative. The teacher should make sure the students are staying on task and guide students to the correct answer.

Modeling: After students have found the product rule and the quotient rule, the teacher should then demonstrate some examples of finding derivatives using the product rule and the quotient rule.

Check for Understanding: The teacher should ask questions during the demonstration to verify students' comprehension.

Guided Practice: Give students the following two problems to work in class and make sure all students get the correct answer. 1) Find $F'(x)$ if $F(x) = (2x^3 + 3)(x^4 - 2x)$. 2) Find $F'(t)$ if $F(t) = \frac{2t}{4+t^2}$. Allow students to work on these independently, and the teacher should circulate through the classroom to view students' work and to be available for any questions.

Closure: Once the lesson is finished, the teacher should review what was learned and the importance and purpose of derivatives. Knowing the importance, purpose, and the history of derivatives allows students to acknowledge math as a human creation. By going through Leibniz's investigations, we see how the derivative is a human creation and might help students remember to use the product rule. Derivatives allow us to investigate concepts such as velocity and acceleration and to find maximums and minimums, which are useful in a variety of subjects such as business and physics.

Evaluation Strategy: Students will be evaluated by their completion of the assigned problems.

Independent Practice: Students will be assigned homework problems to complete independently. These assigned problems should consist of finding derivatives of functions and applications of derivatives.

Resource: *Agnesi to Zeno: Over 100 Vignettes from the History of Math* by Sanderson M. Smith

Appendix E: Pre-Lesson Survey and Data

Pre-Lesson Survey (Geometry)

1. My knowledge of the Pythagorean Theorem is strong.

   strongly agree agree neutral disagree strongly disagree

2. I understand where the Pythagorean Theorem comes from.

   strongly agree agree neutral disagree strongly disagree

3. I understand how to use the Pythagorean Theorem.

   strongly agree agree neutral disagree strongly disagree

4. I think learning about the history of math shows math's purpose and portrays math as a human creation.

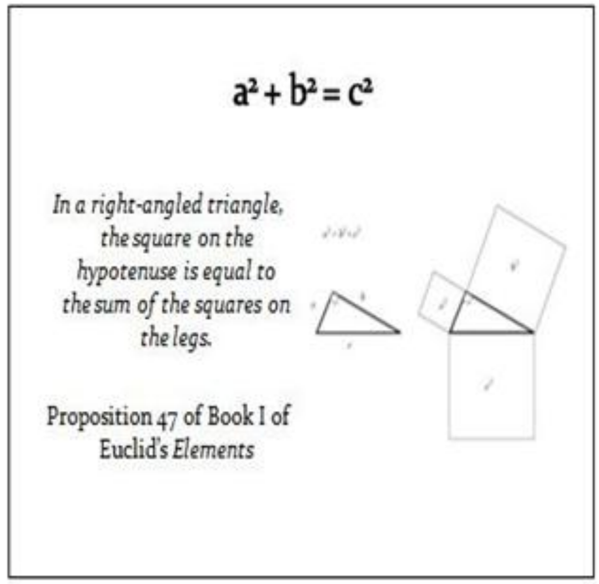
   strongly agree agree neutral disagree strongly disagree

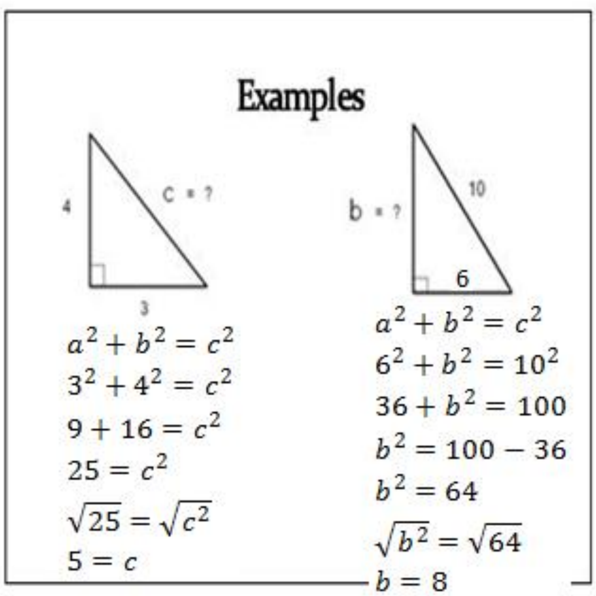
5. Learning the history of Pythagorean Theorem helps me understand it more thoroughly.

   strongly agree agree neutral disagree strongly disagree

Question	Strongly Agree	Agree	Neutral	Disagree	Strongly Disagree
1	5.3%	36.8%	26.3%	15.8%	15.8%
2	0%	15.8%	31.6%	31.6%	21.1%
3	5.3%	57.9%	15.8%	5.3%	15.8%
4	26.3%	52.6%	10.5%	0%	10.5%
5	31.6%	57.9%	10.5%	0%	0%

Appendix F: Pythagorean Theorem Lesson PowerPoint



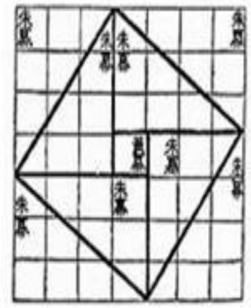


Egyptians

- Cairo Mathematical Papyrus
- Surveyors
- Ladder problem

Chinese

- Oldest known proof



Pythagoras and his followers

- "All things are numbers."
- School of Pythagoras
- Secret society
- Quadrivium
- Strange prohibitions
- Irrational numbers

The Many Proofs

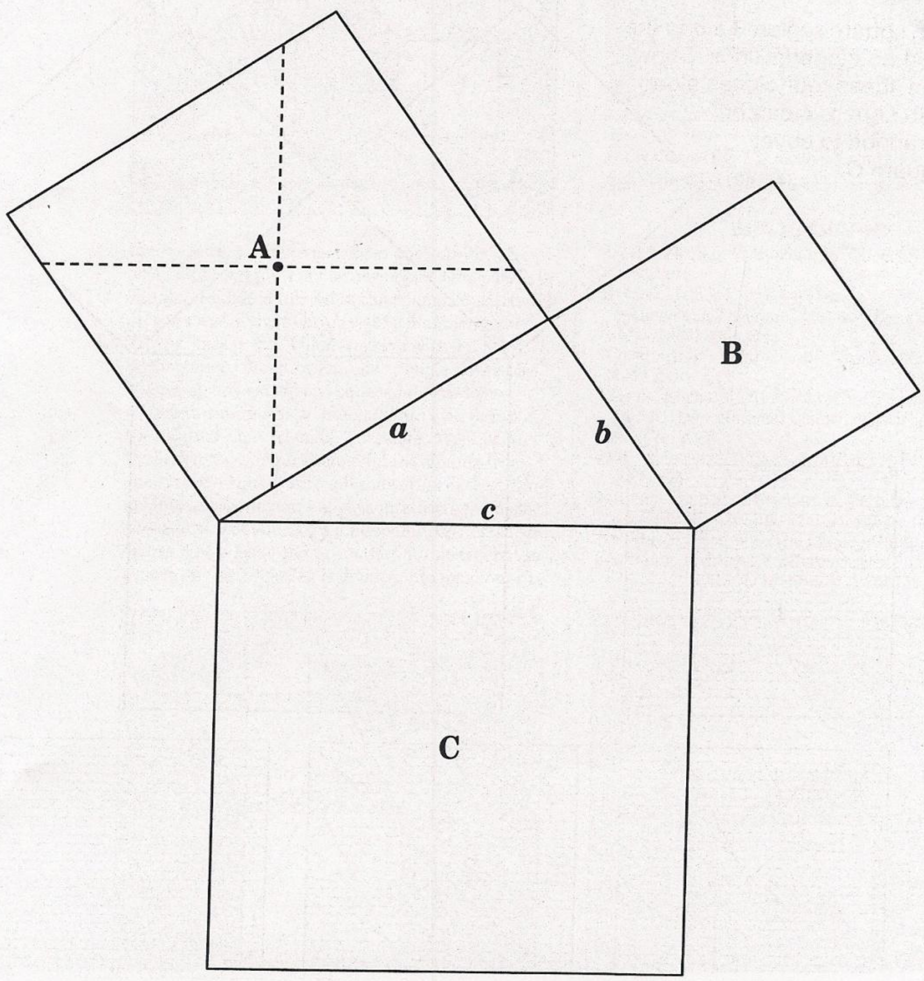
- Elisha Loomis, *The Pythagorean Proposition*
- 367 proofs
- Limitless geometric and algebraic proofs

Appendix G: Pythagorean Theorem Puzzles

PERIGAL'S PYTHAGOREAN PUZZLE

SHEET 2

1. Cut out square 5 on sheet 4. Cover square A to verify that the squares are congruent. Cover square B with square 4 from activity 1 to verify that these pairs of squares are congruent.
2. Cut square 5 along the dashed lines. Show how these four pieces together with square region 4 can be arranged to cover square region C.
3. What relationship appears to be true between $a^2 + b^2$ and c^2? Explain.



From the *Mathematics Teacher*, April 1993

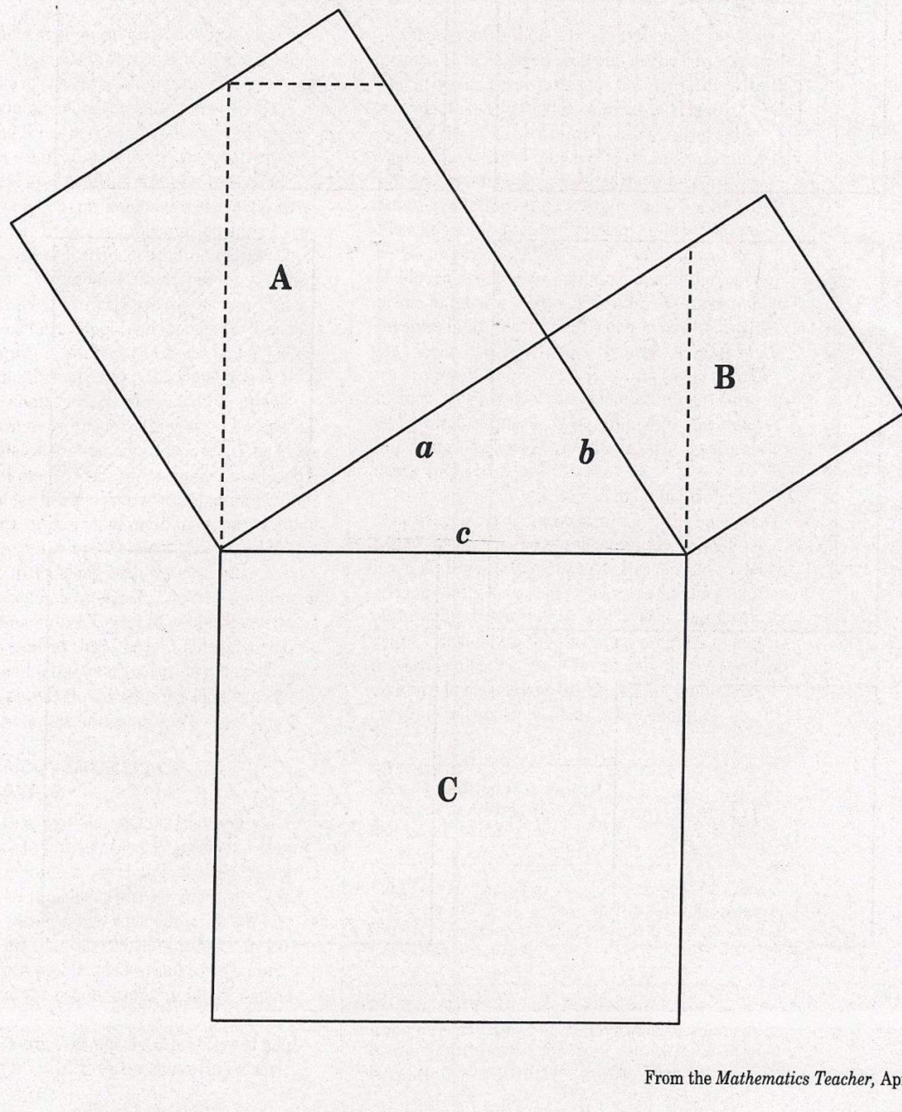
(Continued on page 313)

(Continued from page 308)

LOOMIS'S PYTHAGOREAN PUZZLE

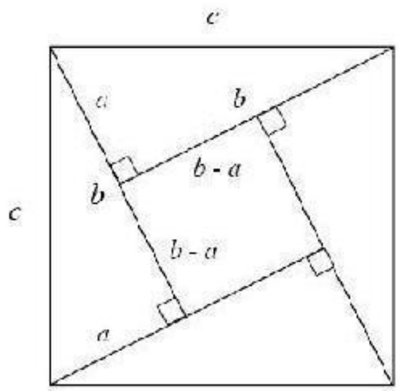
SHEET 3

1. Cut out squares 6 and 7 on sheet 4. Use those squares to cover A and B to verify that each pair of squares is congruent.
2. Cut square regions 6 and 7 along the dashed lines. Show how these five pieces can be arranged to cover square region C.
3. Does $a^2 + b^2$ appear to equal c^2? Explain.



From the *Mathematics Teacher,* April 1993

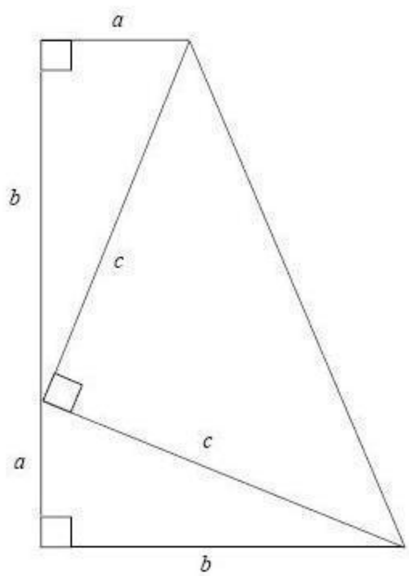
Appendix H: Pythagorean Theorem Proofs



$$c^2 = (b-a)^2 + 4\left(\frac{1}{2}ab\right)$$

$$c^2 = b^2 - 2ab + a^2 + 2ab$$

$$c^2 = b^2 + a^2$$



$$\frac{1}{2}(a+b)(a+b) = 2\left(\frac{1}{2}ab\right) + \frac{1}{2}(c^2)$$

$$\frac{1}{2}(a^2 + 2ab + b^2) = ab + \frac{1}{2}c^2$$

$$\frac{1}{2}a^2 + ab + \frac{1}{2}b^2 = ab + \frac{1}{2}c^2$$

$$\frac{1}{2}a^2 + \frac{1}{2}b^2 = \frac{1}{2}c^2$$

$$a^2 + b^2 = c^2$$

Appendix I: Post-Lesson Survey and Data

Post-Lesson Survey (Geometry)

1. My knowledge of the Pythagorean Theorem is strong.

   strongly agree agree neutral disagree strongly disagree

2. I understand where the Pythagorean Theorem comes from.

   strongly agree agree neutral disagree strongly disagree

3. I understand how to use the Pythagorean Theorem.

   strongly agree agree neutral disagree strongly disagree

4. I think learning about the history of math shows math's purpose and portrays math as a human creation.

   strongly agree agree neutral disagree strongly disagree

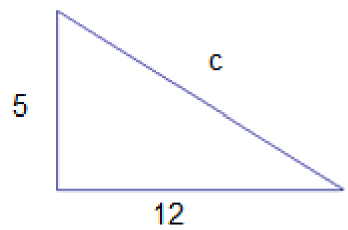
5. Learning the history of Pythagorean Theorem helps me understand it more thoroughly.

   strongly agree agree neutral disagree strongly disagree

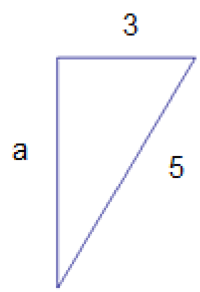
Question	Strongly Agree	Agree	Neutral	Disagree	Strongly Disagree
1	31.6%	26.3%	26.3%	15.8%	0%
2	36.8%	47.4%	15.8%	0%	0%
3	36.8%	36.8%	21.1%	5.3%	0%
4	36.8%	52.6%	10.5%	0%	0%
5	31.6%	47.4%	10.5%	10.5%	0%

Appendix J: Pythagorean Theorem Assessment

1. Find the missing lengths of the following right triangles using the Pythagorean Theorem.



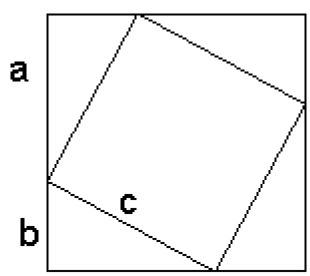
c =



a =

2. What ancient civilization can claim priority to discovering the Pythagorean Theorem?

3. Prove the Pythagorean Theorem using the following figure.
   (Hint: The side of the large square has length a+b.)



4. What was the motto the Pythagoreans lived by?