Chapter 3

Motion in Two and Three Dimensions

3.1 The Important Stuff

3.1.1 Position

In three dimensions, the location of a particle is specified by its **location vector, r**:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{3.1}$$

If during a time interval Δt the position vector of the particle changes from $\mathbf{r}_1$ to $\mathbf{r}_2$, the displacement $\Delta\mathbf{r}$ for that time interval is

$$\Delta\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{3.2}$$

$$= (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k} \tag{3.3}$$

3.1.2 Velocity

If a particle moves through a displacement $\Delta\mathbf{r}$ in a time interval Δt then its average velocity for that interval is

$$\overline{v} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\mathbf{i} + \frac{\Delta y}{\Delta t}\mathbf{j} + \frac{\Delta z}{\Delta t}\mathbf{k} \tag{3.4}$$

As before, a more interesting quantity is the *instantaneous* velocity $\mathbf{v}$, which is the limit of the average velocity when we shrink the time interval Δt to zero. It is the time derivative of the position vector $\mathbf{r}$:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \tag{3.5}$$

$$= \frac{d}{dt}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \tag{3.6}$$

$$= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \tag{3.7}$$

can be written:

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} \tag{3.8}$$

where

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \qquad v_z = \frac{dz}{dt} \tag{3.9}$$

The instantaneous velocity $\mathbf{v}$ of a particle is always tangent to the path of the particle.

3.1.3 Acceleration

If a particle's velocity changes by $\Delta\mathbf{v}$ in a time period Δt, the average acceleration $\overline{\mathbf{a}}$ for that period is

$$\overline{\mathbf{a}} = \frac{\Delta\mathbf{v}}{\Delta t} = \frac{\Delta v_x}{\Delta t}\mathbf{i} + \frac{\Delta v_y}{\Delta t}\mathbf{j} + \frac{\Delta v_z}{\Delta t}\mathbf{k} \tag{3.10}$$

but a much more interesting quantity is the result of shrinking the period Δt to zero, which gives us the instantaneous acceleration, $\mathbf{a}$. It is the time derivative of the velocity vector $\mathbf{v}$:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \tag{3.11}$$

$$= \frac{d}{dt}(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}) \tag{3.12}$$

$$= \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k} \tag{3.13}$$

which can be written:

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \tag{3.14}$$

where

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \qquad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \qquad a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2} \tag{3.15}$$

3.1.4 Constant Acceleration in Two Dimensions

When the acceleration $\mathbf{a}$ (for motion in two dimensions) is constant we have two sets of equations to describe the x and y coordinates, each of which is similar to the equations in Chapter 2. (Eqs. 2.6—2.9.) In the following, motion of the particle begins at $t = 0$; the initial position of the particle is given by

$$\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j}$$

and its initial velocity is given by

$$\mathbf{v}_0 = v_{0x}\mathbf{i} + v_{0y}\mathbf{j}$$

and the vector $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j}$ is *constant.*

$$v_x = v_{0x} + a_x t \qquad v_y = v_{0y} + a_y t \tag{3.16}$$

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2 \qquad y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2 \tag{3.17}$$

$$v_x^2 = v_{0x}^2 + 2a_x(x - x_0) \qquad v_y^2 = v_{0y}^2 + 2a_y(y - y_0) \tag{3.18}$$

$$x = x_0 + \tfrac{1}{2}(v_{0x} + v_x)t \qquad y = y_0 + \tfrac{1}{2}(v_{0y} + v_y)t \tag{3.19}$$

Though the equations in each pair have the same *form* they are not identical because the components of $\mathbf{r}_0$, $\mathbf{v}_0$ and $\mathbf{a}$ are not the same.

3.1.5 Projectile Motion

When a particle moves in a vertical plane during free–fall its acceleration is constant; the acceleration has magnitude $9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$ and is directed downward. If its coordinates are given by a horizontal x axis and a vertical y axis which is directed upward, then the acceleration of the **projectile** is

$$a_x = 0 \qquad a_y = -9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2} = -g \tag{3.20}$$

For a projectile, the horizontal acceleration a_x is zero!!!

Projectile motion is a special case of constant acceleration, so we simply use Eqs. 3.16–3.19, with the proper values of a_x and a_y.

3.1.6 Uniform Circular Motion

When a particle is moving in a circular path (or part of one) at *constant speed* we say that the particle is in **uniform circular motion**. Even though the speed is not changing, *the particle is accelerating* because its velocity $\mathbf{v}$ is changing *direction*.

The acceleration of the particle is directed toward the center of the circle and has magnitude

$$a = \frac{v^2}{r} \tag{3.21}$$

where r is the radius of the circular path and v is the (constant) speed of the particle. Because of the direction of the acceleration (i.e. toward the center), we say that a particle in uniform circular motion has a **centripetal acceleration**.

If the particle repeatedly makes a complete circular path, then it is useful to talk about the time T that it takes for the particle to make one complete trip around the circle. This is called the **period** of the motion. The period is related to the speed of the particle and radius of the circle by:

$$T = \frac{2\pi r}{v} \tag{3.22}$$

3.1.7 Relative Motion

The velocity of a particle depends on who is doing the measuring; as we see later on it is perfectly valid to consider "moving" observers who carry their own clocks and coordinate systems with them, i.e. they make measurements according to their own **reference frame**; that is to say, a set of Cartesian coordinates which may be in motion with respect to another set of coordinates. Here we will assume that the axes in the different system remain parallel to one another; that is, one system can move (translate) but not *rotate* with respect to another one.

Suppose observers in frames A and B measure the position of a point P. Then then if we have the definitions:

$$\mathbf{r}_{PA} = \text{ position of } P \text{ as measured by } A$$

$$\mathbf{r}_{PB} = \text{ position of } P \text{ as measured by } B$$

$$\mathbf{r}_{BA} = \text{ position of } B\text{'s origin, as measured by } A$$

with **v**'s and **a**'s standing for the appropriate time derivatives, then we have the relations:

$$\mathbf{r}_{PA} = \mathbf{r}_{PB} + \mathbf{r}_{BA} \tag{3.23}$$

$$\mathbf{v}_{PA} = \mathbf{v}_{PB} + \mathbf{v}_{BA} \tag{3.24}$$

For the purposes of doing physics, it is important to consider reference frames which move at *constant velocity* with respect to one another; for these cases, $\mathbf{v}_{BA} = 0$ and then we find that point P has the same acceleration in these reference frames:

$$\mathbf{a}_{PA} = \mathbf{a}_{PB}$$

Newton's Laws (next chapter!) apply to such a set of **inertial reference frames**. Observers in each of these frames agree on the value of a particle's acceleration.

Though the above rules for translation between reference frames seem very reasonable, it was the great achievement of Einstein with his theory of **Special Relativity** to understand the more subtle ways that we must relate measured quantities between reference frames. The trouble comes about because time (t) is *not* the same absolute quantity among the different frames.

Among other places, Eq. 3.24 is used in problems where an object like a plane or boat has a known velocity in the frame of (with respect to) a medium like air or water which *itself* is moving with respect to the stationary ground; we can then find the velocity of the plane or boat with respect to the *ground* from the *vector sum* in Eq. 3.24.

3.2 Worked Examples

3.2.1 Velocity

1. The position of an electron is given by $\mathbf{r} = 3.0t\mathbf{i} - 4.0t^2\mathbf{j} + 2.0\mathbf{k}$ (where t is in seconds and the coefficients have the proper units for r to be in meters). (a) What is $\mathbf{v}(t)$ for the electron? (b) In unit–vector notation, what is v at $t = 2.0\,\mathrm{s}$? (c) What are the magnitude and direction of v just then? [HRW5 4-9]

(a) The velocity vector **v** is the time–derivative of the position vector **r**:

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(3.0t\mathbf{i} - 4.0t^2\mathbf{j} + 2.0\mathbf{k}) \\ &= 3.0\mathbf{i} - 8.0t\mathbf{j} \end{aligned}$$

where we mean that when t is in seconds, **v** is given in $\frac{\mathrm{m}}{\mathrm{s}}$.

(b) At $t = 2.00\,\mathrm{s}$, the value of $\mathbf{v}$ is

$$\mathbf{v}(t = 2.00\,\mathrm{s}) = 3.0\mathbf{i} - (8.0)(2.0)\mathbf{j} = 3.0\mathbf{i} - 16.\mathbf{j}$$

that is, the velocity is $(3.0\mathbf{i} - 16.\mathbf{j})\,\frac{\mathrm{m}}{\mathrm{s}}$.

(c) Using our answer from (b), at $t = 2.00\,\mathrm{s}$ the magnitude of $\mathbf{v}$ is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{(3.00\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + (-16.\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + (0)^2} = 16.\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

we note that the velocity vector lies in the xy plane (even though this is a three–dimensional problem!) so that we can express its direction with a single angle, the usual angle θ measured anti-clockwise in the xy plane from the x axis. For this angle we get:

$$\tan\theta = \frac{v_y}{v_x} = -5.33 \qquad \Longrightarrow \qquad \theta = \tan^{-1}(-5.33) = -79^\circ\ .$$

When we take the inverse tangent, we should always check and see if we have chosen the right quadrant for θ. In this case -79° *is* correct since v_y is negative and v_x is positive.

3.2.2 Acceleration

---

2. A particle moves so that its position as a function of time in SI units is $\mathbf{r} = \mathbf{i} + 4t^2\mathbf{j} + t\mathbf{k}$. Write expressions for (a) its velocity and (b) its acceleration as functions of time. [HRW5 4-11]

(a) To clarify matters, what we mean here is that when we use the numerical value of t in *seconds*, we will get the values of $\mathbf{r}$ in *meters*. Since the velocity vector is the time–derivative of the position vector $\mathbf{r}$, we have:

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ &= \frac{d}{dt}(\mathbf{i} + 4t^2\mathbf{j} + t\mathbf{k}) \\ &= 0\mathbf{i} + 8t\mathbf{j} + \mathbf{k}\end{aligned}$$

That is, $\mathbf{v} = 8t\mathbf{j} + \mathbf{k}$. Here, we mean that when we use the numerical value of t in seconds, we will get the value of $\mathbf{v}$ in $\frac{\mathrm{m}}{\mathrm{s}}$.

(b) The acceleration $\mathbf{a}$ is the time–derivative of $\mathbf{v}$, so using our result from part (a) we have:

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} \\ &= \frac{d}{dt}(8t\mathbf{j} + \mathbf{k}) \\ &= 8\mathbf{j}\end{aligned}$$

So $\mathbf{a} = 8\mathbf{j}$, where we mean that the value of $\mathbf{a}$ is in units of $\frac{\mathrm{m}}{\mathrm{s}^2}$. In fact, we should really include the units *here* and write:

$$\mathbf{a} = \left(8\,\tfrac{\mathrm{m}}{\mathrm{s}^2}\right)\mathbf{j}$$

---

3. A particle moving with an initial velocity $\mathbf{v} = (50\,\frac{\mathrm{m}}{\mathrm{s}})\mathbf{j}$ undergoes an acceleration $\mathbf{a} = [35\,\mathrm{m/s^2} + (2\,\mathrm{m/s^5})t^3]\mathbf{i} + [4\,\mathrm{m/s^2} - (1\,\mathrm{m/s^4})t^2]\mathbf{j}$. What are the particle's position and velocity after $3.0\,\mathrm{s}$, assuming that it starts at the origin? [FGT2 3-20]

In the problem we are given the acceleration at *all* times, the *initial* velocity and also the *initial* position. We know that at $t = 0$, the velocity components are $v_x = 0$ and $v_y = 50\,\frac{\mathrm{m}}{\mathrm{s}}$ and the coordinates are $x = 0$ and $y = 0$.

From the acceleration $\mathbf{a}$ we do know *something* about the velocity. Since the acceleration is the time derivative of the velocity:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \ ,$$

the velocity is the *anti-derivative* (or "indefinite integral", "primitive"...) of the acceleration. Having learned our calculus well, we immediately write:

$$\mathbf{v} = \left[35t + \frac{1}{2}t^4 + C_1\right]\mathbf{i} + \left[4t - \frac{1}{3}t^3 + C_2\right]\mathbf{j}$$

Here, for simplicity, I have omitted the units that are supposed to go with the coefficients. (I'm not supposed to do that!) Just keep in mind that time is supposed to be in *seconds*, length is in *meters*...

Of course, when we do the integration, we get constants C_1 and C_2 which (so far) have not been determined. We can determine them using the rest of the information in the problem. Since $v_x = 0$ at $t = 0$ we get:

$$35(0) + \tfrac{1}{2}(0)^4 + C_1 = 0 \qquad \Longrightarrow \qquad C_1 = 0$$

and

$$4(0) - \tfrac{1}{3}(0)^3 + C_2 = 50 \qquad \Longrightarrow \qquad C_2 = 50$$

so the velocity as a function of time is

$$\mathbf{v} = \left[35t + \frac{1}{2}t^4\right]\mathbf{i} + \left[4t - \frac{1}{3}t^3 + 50\right]\mathbf{j}$$

where t is in seconds and the result is in $\frac{\mathrm{m}}{\mathrm{s}}$.

We can find $\mathbf{r}$ as a function of time in the same way. Since

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

then **r** is the anti-derivative of **v**. We get:

$$\mathbf{r} = \left[\frac{35}{2}t^2 + \frac{1}{10}t^5 + C_3\right]\mathbf{i} + \left[2t^2 - \frac{1}{12}t^4 + 50t + C_4\right]\mathbf{j}$$

and once again we need to solve for the constants. $x = 0$ at $t = 0$, so

$$\frac{35}{2}(0)^2 + \frac{1}{10}(0)^5 + C_3 = 0 \qquad \Longrightarrow \qquad C_3 = 0$$

and $y = 0$ at $t = 0$, so

$$2(0)^2 - \frac{1}{12}(0)^4 + 50(0) + C_4 = 0 \qquad \Longrightarrow \qquad C_4 = 0$$

and so **r** is fully determined:

$$\mathbf{r} = \left[\frac{35}{2}t^2 + \frac{1}{10}t^5\right]\mathbf{i} + \left[2t^2 - \frac{1}{12}t^4 + 50t\right]\mathbf{j}$$

Now we can answer the questions.

We want to know the value of **r** (the particle's position) at $t = 3.0\,\mathrm{s}$. Just plug in!

$$x(t = 3.0\,\mathrm{s}) = \frac{35}{2}(3.0)^2 + \frac{1}{10}(3.0)^5 = 181\,\mathrm{m}$$

and

$$y(t = 3.0\,\mathrm{s}) = 2(3.0)^2 - \frac{1}{12}(3.0)^4 + 50(3.0) = 161\,\mathrm{m}\ .$$

The components of the velocity at $t = 3.0\,\mathrm{s}$ are

$$v_x(t = 3.0\,\mathrm{s}) = 35(3.0) + \tfrac{1}{2}(3.0)^4 = 146\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

and

$$v_y(t = 3.0\,\mathrm{s}) = 4(3.0) - \tfrac{1}{3}(3.0)^3 + 50 = 53\,\tfrac{\mathrm{m}}{\mathrm{s}}\ .$$

Here we have been careful to include the proper (SI) units in the final answers because coordinates and velocities must have *units*.

3.2.3 Constant Acceleration in Two Dimensions

---

4. A fish swimming in a horizontal plane has a velocity $\mathbf{v}_0 = (4.0\mathbf{i} + 1.0\mathbf{j})\,\frac{\mathrm{m}}{\mathrm{s}}$ at a point in the ocean whose position vector is $\mathbf{r}_0 = (10.0\mathbf{i} - 4.0\mathbf{j})\,\mathrm{m}$ relative to a stationary rock at the shore. After the fish swims with constant acceleration for $20.0\,\mathrm{s}$, its velocity is $\mathbf{v} = (20.0\mathbf{i} - 5.0\mathbf{j})\,\frac{\mathrm{m}}{\mathrm{s}}$. (a) What are the components of the acceleration? (b) What is the direction of the acceleration with respect to the fixed x axis? (c) Where is the fish at $t = 25\,\mathrm{s}$ and in what direction is it moving?

[Ser4 4-7]

(a) Since we are given that the acceleration is *constant*, we can use Eqs. 3.16:

$$v_x = v_{0x} + a_x t \qquad\qquad v_y = v_{0y} + a_y t$$

to get:

$$a_x = \frac{v_x - v_{0x}}{t} = \frac{(20.0\,\frac{\mathrm{m}}{\mathrm{s}} - 4.0\,\frac{\mathrm{m}}{\mathrm{s}})}{20.0\,\mathrm{s}} = 0.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

and

$$a_y = \frac{v_y - v_{0y}}{t} = \frac{(-5.0\,\frac{\mathrm{m}}{\mathrm{s}} - 1.0\,\frac{\mathrm{m}}{\mathrm{s}})}{20.0\,\mathrm{s}} = -0.30\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

and the acceleration *vector* of the fish is

$$\mathbf{a} = (0.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})\mathbf{i} - (0.30\,\tfrac{\mathrm{m}}{\mathrm{s}^2})\mathbf{j}\ .$$

(b) With the angle θ measured counterclockwise from the $+x$ axis, the direction of the acceleration $\mathbf{a}$ is:

$$\tan\theta = \frac{a_y}{a_x} = \frac{-0.30}{0.80} = -0.375$$

A calculator gives us:

$$\theta = \tan^{-1}(-0.375) = -20.6^\circ$$

Since the y component of the acceleration is negative, this angle *is* in the proper quadrant. The direction of the acceleration is given by $\theta = -20.6^\circ$. (The same as $\theta = 360^\circ - 20.6^\circ = 339.4^\circ$.

(c) We can use Eq. 3.17 to find the values of x and y at $t = 25\,\mathrm{s}$:

$$\begin{aligned} x &= x_0 + v_{0x} + \tfrac{1}{2}a_x t^2 \\ &= 10\,\mathrm{m} + 4.0\,\tfrac{\mathrm{m}}{\mathrm{s}}(25\,\mathrm{s}) + \tfrac{1}{2}(0.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(25\,\mathrm{s})^2 \\ &= 360\,\mathrm{m} \end{aligned}$$

and

$$\begin{aligned} y &= y_0 + v_{0y} + \tfrac{1}{2}a_y t^2 \\ &= -4.0\,\mathrm{m} + 1.0\,\tfrac{\mathrm{m}}{\mathrm{s}}(25\,\mathrm{s}) + \tfrac{1}{2}(-0.30\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(25\,\mathrm{s})^2 \\ &= -72.8\,\mathrm{m} \end{aligned}$$

At $t = 25\,\mathrm{s}$ the velocity components of the fish are given by:

$$\begin{aligned} v_x &= v_{0x} + a_x t \\ &= 4.0\,\tfrac{\mathrm{m}}{\mathrm{s}} + (0.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(25\,\mathrm{s}) = 24\,\tfrac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

and

$$\begin{aligned} v_y &= v_{0y} + a_y t \\ &= 1.0\,\tfrac{\mathrm{m}}{\mathrm{s}} + (-0.30\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(25\,\mathrm{s}) = -6.5\,\tfrac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

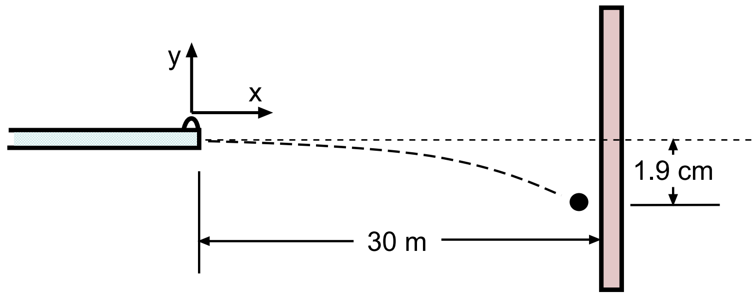

Figure 3.1: Bullet hits target 1.9 cm below the aiming point.

so that at that time the speed of the fish is

$$\begin{aligned} v &= \sqrt{v_x^2 + v_y^2} \\ &= \sqrt{(24\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + (-6.5\,\tfrac{\mathrm{m}}{\mathrm{s}})^2} = 24.9\,\tfrac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

and the direction of its motion θ is found from:

$$\tan\theta = \frac{v_y}{v_x} = \frac{-6.5}{24} = -0.271$$

so that

$$\theta = -15.2^\circ \ .$$

Again, since v_y is negative and v_x is positive, this is the correct choice for θ. So the direction of the fish's motion is -15.2° from the $+x$ axis.

3.2.4 Projectile Motion

---

5. A rifle is aimed horizontally at a target 30 m away. The bullet hits the target 1.9 cm below the aiming point. (a) What is the bullet's time of flight? (b) What is the muzzle velocity? [HRW5 4-19]

(a) First, we define our coordinates. I will use the coordinate system indicated in Fig. 3.1, where the origin is placed at the tip of the gun. Then we have $x_0 = 0$ and $y_0 = 0$. We also know the acceleration:

$$a_x = 0 \qquad \text{and} \qquad a_y = -9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2} = -g$$

What else do we know? The gun is fired *horizontally* so that $v_{0y} = 0$, but we don't know v_{0x}. We don't know the time of flight but we do know that when x has the value 30 m then y has the value -1.9×10^{-2} m. (Minus!)

Our equation for the y coordinate is

$$\begin{aligned} y &= y_0 + y_{0y}t + \tfrac{1}{2}a_y t^2 \\ &= 0 + 0 + \tfrac{1}{2}(-g)t^2 \\ &= -\tfrac{1}{2}gt^2 \end{aligned}$$

We can now ask: "At what time t does y equal $-1.9 \times 10^{-2}\,\mathrm{m}$?" . Substitute $y = -1.9 \times 10^{-2}\,\mathrm{m}$ and solve:

$$t^2 = -\frac{2y}{g} = -\frac{2(-1.9 \times 10^{-2}\,\mathrm{m})}{9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}} = 3.9 \times 10^{-3}\,\mathrm{s}^2$$

which gives:

$$t = 6.2 \times 10^{-2}\,\mathrm{s}$$

Since this is the time of impact with the target, the time of flight of the bullet is $t = 6.2 \times 10^{-2}\,\mathrm{s}$.

(b) The equation for $x-$motion is

$$\begin{aligned} x &= x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2 \\ &= 0 + v_{0x}t + 0 \\ &= v_{0x}t \end{aligned}$$

From part (a) we know that when $t = 6.2 \times 10^{-2}\,\mathrm{s}$ then $x = 30\,\mathrm{m}$. This allows us to solve for v_{0x}:

$$v_{0x} = \frac{x}{t} = \frac{30\,\mathrm{m}}{6.2 \times 10^{-2}\,\mathrm{s}} = 480\,\frac{\mathrm{m}}{\mathrm{s}}$$

The muzzle velocity of the bullet is $480\,\frac{\mathrm{m}}{\mathrm{s}}$.

---

6. In a local bar, a customer slides an empty beer mug on the counter for a refill. The bartender does not see the mug, which slides off the counter and strikes the floor $1.40\,\mathrm{m}$ from the base of the counter. If the height of the counter is $0.860\,\mathrm{m}$, (a) with what speed did the mug leave the counter and (b) what was the direction of the mug's velocity just before it hit the floor? [Ser4 4-11]

(a) The motion of the beer mug is shown in Fig. 3.2(a). We choose the origin of our xy coordinate system as being at the point where the mug leaves the counter. So the mug's initial coordinates for its flight are $x_0 = 0$, $y_0 = 0$.

At the very beginning of its motion through the air, the velocity of the mug is *horizontal*. (This is because its velocity was horizontal all the time it was sliding on the counter.) So we know that $v_{0y} = 0$ but we don't know the value of v_{0x}. (In fact, that's what we're trying to figure out!)

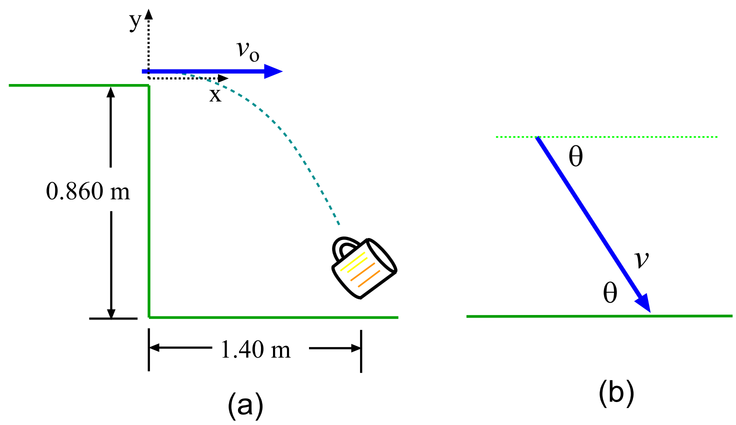

Figure 3.2: (a) Beer mug slides off counter and strikes floor! (b) Velocity vector of the beer mug at the time of impact.

We might begin by finding the time t at which the mug hit the floor. This is the time t at which $y = -0.860\,\mathrm{m}$ (recall how we chose the coordinates!), and we will need the y equation of motion for this; since $v_{0y} = 0$ and $a_y = -g$, we get:

$$y = v_{0y}t + \tfrac{1}{2}a_y t^2 = -\tfrac{1}{2}gt^2$$

So we solve

$$-0.860\,\mathrm{m} = -\tfrac{1}{2}gt^2$$

which gives

$$t^2 = \frac{2(0.860\,\mathrm{m})}{g} = \frac{2(0.860\,\mathrm{m})}{(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 0.176\,\mathrm{s}^2$$

so then

$$t = 0.419\,\mathrm{s}$$

is the time of impact.

To find v_{0x} we consider the x equation of motion; $x_0 = 0$ and $a_x = 0$, so we have

$$x = v_{0x}t \ .$$

At $t = 0.419\,\mathrm{s}$ we know that the x coordinate was equal to $1.40\,\mathrm{m}$. So

$$1.40\,\mathrm{m} = v_{0x}(0.419\,\mathrm{s})$$

Solve for v_{0x}:

$$v_{0x} = \frac{1.40\,\mathrm{m}}{0.419\,\mathrm{s}} = 3.34\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

which tells us that the initial speed of the mug was $v_0 = 3.34\,\frac{\mathrm{m}}{\mathrm{s}}$.

(b) We want to find the components of the mug's velocity at the time of impact, that is, at $t = 0.419\,\mathrm{s}$. Substitute into our expressions for v_x and v_y:

$$v_x = v_{0x} + a_x t = v_{0x} = 3.34\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

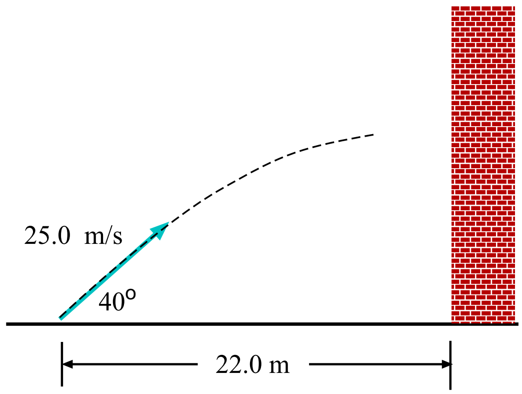

Figure 3.3: Ball is thrown toward wall at 40° above horizontal, in Example 7.

and

$$v_y = v_{0y} + a_y t = 0 + (-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(0.419\,\mathrm{s}) = -4.11\,\tfrac{\mathrm{m}}{\mathrm{s}} \ .$$

So at the time of impact, the *speed* of the mug was

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(3.34\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 + (-4.11\,\tfrac{\mathrm{m}}{\mathrm{s}})^2} = 5.29\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

and, if as in Fig. 3.2(b) we let θ be the angle *below the horizontal* at which the velocity vector is pointing, we see that

$$\tan\theta = \frac{4.11}{3.34} = 1.23 \qquad \Longrightarrow \qquad \theta = \tan^{-1}(1.23) = 50.9^\circ \ .$$

At the time of impact, the velocity of the mug was directed at 50.9° below the horizontal.

---

7. You throw a ball with a speed of $25.0\,\frac{\mathrm{m}}{\mathrm{s}}$ at an angle of 40.0° above the horizontal directly toward a wall, as shown in Fig. 3.3. The wall is $22.0\,\mathrm{m}$ from the release point of the ball. (a) How long does the ball take to reach the wall? (b) How far above the release point does the ball hit the wall? (c) What are the horizontal and vertical components of its velocity as it hits the wall? (d) When it hits, has it passed the highest point on its trajectory? [HRW5 4-28]

(a) We will use a coordinate system which has its origin at the point of firing, which we take to be at ground level.

What is the mathematical condition which determines when the ball hits the wall? It is when the x coordinate of the ball is equal to 22.0 m. Then let's write out the $x-$equation of motion for the ball. The ball's initial $x-$ velocity is

$$v_{0x} = v_0 \cos\theta_0 = (25.0\,\tfrac{\mathrm{m}}{\mathrm{s}})\cos 40.0^\circ = 19.2\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

and of course $a_x = 0$, so that the x motion is given by

$$x = x_0 + v_{ox}t + \tfrac{1}{2}a_x t^2 = 19.2\,\tfrac{\mathrm{m}}{\mathrm{s}}t$$

We solve for the time at which $x = 22.0\,\mathrm{m}$:

$$x = 19.2\,\tfrac{\mathrm{m}}{\mathrm{s}}t = 22.0\,\mathrm{m} \qquad \Longrightarrow \qquad t = \frac{22.0\,\mathrm{m}}{19.2\,\frac{\mathrm{m}}{\mathrm{s}}} = 1.15\,\mathrm{s}$$

The ball hits the wall 1.15 s after being thrown.

(b) We will be able to answer this question if we can find the y *coordinate* of the ball at the time that it hits the wall, namely at $t = 1.15\,\mathrm{s}$.

We need the y equation of motion. The initial y velocity of the ball is

$$v_{0y} = v_0 \sin\theta_0 = \left(25.0\,\tfrac{\mathrm{m}}{\mathrm{s}}\right)\sin 40.0^\circ = 16.1\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

and the y acceleration of the ball is $a_y = -g$ giving:

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2 = \left(16.1\,\tfrac{\mathrm{m}}{\mathrm{s}}\right)t - \tfrac{1}{2}gt^2$$

which we use to find the y coordinate at $t = 1.15\,\mathrm{s}$:

$$y = (16.1\,\tfrac{\mathrm{m}}{\mathrm{s}})(1.15\,\mathrm{s}) - \tfrac{1}{2}(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(1.15\,\mathrm{s})^2 = 12.0\,\mathrm{m}$$

which tells us that the ball hits the wall at 12.0 m above the ground level (above the release point).

(c) The x and y components of the balls's velocity at the time of impact, namely at $t = 1.15\,\mathrm{s}$ are found from Eqs. 3.16:

$$v_x = v_{0x} + a_x t = 19.2\,\tfrac{\mathrm{m}}{\mathrm{s}} + 0 = 19.2\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

and

$$v_y = v_{0y} + a_y t = 16.1\,\tfrac{\mathrm{m}}{\mathrm{s}} + (-9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(1.15\,\mathrm{s}) = +4.83\,\tfrac{\mathrm{m}}{\mathrm{s}}\ .$$

(d) Has the ball already passed the highest point on its trajectory? Suppose the ball was on its way *downward* when it struck the wall. Then the y component of the velocity would be *negative*, since it is always decreasing and at the trajectory's highest point it is zero. (Of course, the x component of the velocity stays the same while the ball is in flight.) Here we see that the y component of the ball's velocity is still *positive* at the time of impact. So the ball was still climbing when it hit the wall; it had *not* reached the highest point of its (free) trajectory.

---

8. The launching speed of a certain projectile is five times the speed it has at its maximum height. Calculate the elevation angle at launching. [HRW5 4-32]

We make a diagram of the projectile's motion in Fig. 3.4. The launch it speed is v_0, and the projectile is launched at an angle θ_0 upward from the horizontal.

We might start this problem by solving for the time it takes the projectile to get to maximum height, but we can note that at maximum height, there is no y velocity component, and

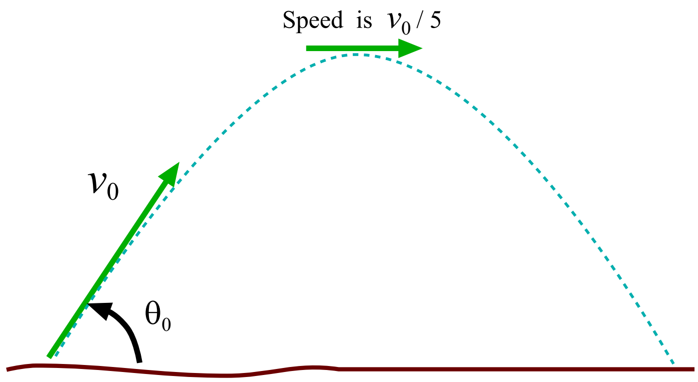

Figure 3.4: Motion of projectile in Example 8.

the x velocity component is *the same as it was when the projectile was launched*. Therefore at maximum height the velocity components are

$$v_x = v_0 \cos\theta_0 \qquad \text{and} \qquad v_y = 0$$

and so the speed of the projectile at maximum height is $v_0 \cos\theta_0$.

Now, we are told that the launching speed (v_0) is five times the speed at maximum height. This gives us:

$$v_0 = 5 v_0 \cos\theta_0 \qquad \Longrightarrow \qquad \cos\theta_0 = \frac{1}{5}$$

which has the solution

$$\theta_0 = 78.5^\circ$$

So the elevation angle at launching is $\theta_0 = 78.5^\circ$.

---

9. A projectile is launched from ground level with speed v_0 at an angle of θ_0 above the horizontal. Find: (a) the maximum height H attained by the projectile, and (b) the distance from the starting point at which the projectile strikes the ground; this is called the range R of the projectile.

Comment: This problem is worked in virtually every physics text, and it is sometimes simply called "The Projectile Problem". I include it in this book for the sake of completeness and so that we can use the results if we need them later on. I do *not* treat it as part of the fundamental material of this chapter because it is a very particular application of free–fall motion. In this problem, the projectile impacts at the *same height* as the one from which it started, and that is *not* always the case. We must think about all projectile problems *individually* and not rely on simple formulae to plug numbers into!

The path of the projectile is shown in Fig. 3.5. The initial coordinates of the projectile are

$$x_0 = 0 \qquad \text{and} \qquad y_0 = 0 \ ,$$

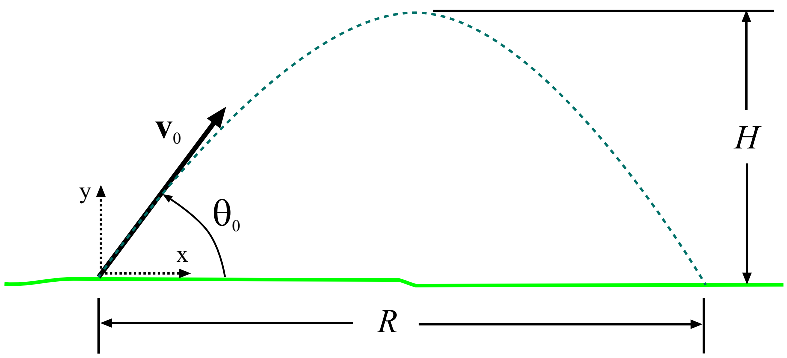

Figure 3.5: The common projectile problem; projectile is shot from ground level with speed v_0 and angle θ_0 above the horizontal.

the components of the initial velocity are

$$v_{0x} = v_0 \cos\theta_0 \qquad \text{and} \qquad v_{0y} = v_0 \sin\theta_0$$

and of course the (constant) acceleration of the projectile is

$$a_x = 0 \qquad \text{and} \qquad a_y = -g = -9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2}$$

Then our equations for $x(t)$, $v_x(t)$, $y(t)$ and $v_y(t)$ are

$$\begin{aligned} v_x &= v_0 \cos\theta_0 \\ x &= v_0 \cos\theta_0\, t \\ v_y &= v_0 \sin\theta_0 - gt \\ y &= v_0 \sin\theta_0\, t - \tfrac{1}{2} g t^2 \end{aligned}$$

(a) What does it mean for the projectile to get to "maximum height"? This is when it is neither increasing in height (rising) nor decreasing in height (falling); the vertical component of the velocity at this point is *zero*. At this particular time then,

$$v_y = v_0 \sin\theta_0 - gt = 0$$

so solving this equation for t, the projectile reaches maximum height at

$$t = \frac{v_0 \sin\theta_0}{g} \ .$$

How high is the projectile at this time? To answer this, substitute this value of t into the equation for y and get:

$$\begin{aligned} y &= v_0 \sin\theta_0 \left(\frac{v_0 \sin\theta_0}{g}\right) - \tfrac{1}{2} g \left(\frac{v_0 \sin\theta_0}{g}\right)^2 \\ &= \frac{v_0^2 \sin^2\theta_0}{g} - \frac{v_0^2 \sin^2\theta_0}{2g} \\ &= \frac{v_0^2 \sin^2\theta_0}{2g} \end{aligned}$$

This is the maximum height attained by the projectile:

$$H = \frac{v_0^2 \sin^2\theta_0}{2g}$$

(b) What is the *mathematical* condition for when the projectile strikes the ground (since that is how we will find the range R)? We know that at this point, the projectile's y coordinate is zero:

$$y = v_0 \sin\theta_0\, t - \tfrac{1}{2}gt^2 = 0$$

We want to solve this equation for t; we can factor out t in this expression to get:

$$t(v_0 \sin\theta_0 - \tfrac{1}{2}gt) = 0$$

which has two solutions:

$$t = 0 \qquad \text{and} \qquad t = \frac{2v_0\sin\theta_0}{g}$$

The first of these is just the time when the projectile was fired; yes, y *was* equal to zero then, but that's not what we want! The time at which the projectile strikes the ground is

$$t = \frac{2v_0\sin\theta_0}{g}\ .$$

We want to find the value of x at the time of impact. Substituting this value of t into our equation for $x(t)$, we find:

$$\begin{aligned} x &= v_0\cos\theta_0\left(\frac{2v_0\sin\theta_0}{g}\right) \\ &= \frac{2v_0^2\sin\theta_0\cos\theta_0}{g} \end{aligned}$$

This value of x is the range R of the projectile.

We can make this result a little simpler by recalling the trig relation:

$$\sin 2\theta_0 = 2\sin\theta_0\cos\theta_0\ .$$

Using this in our result for the range gives:

$$R = \frac{2v_0^2\sin\theta_0\cos\theta_0}{g} = \frac{v_0^2\sin 2\theta_0}{g}$$

---

10. A projectile is fired in such a way that its horizontal range is equal to three times its maximum height. What is the angle of projection? [Ser4 4-23]

Now, this problem *does* deal with a projectile which starts and ends its flight at the same height, just as we calculated in the previous example. So we *can* use the results for the range R and maximum height H that we found there.

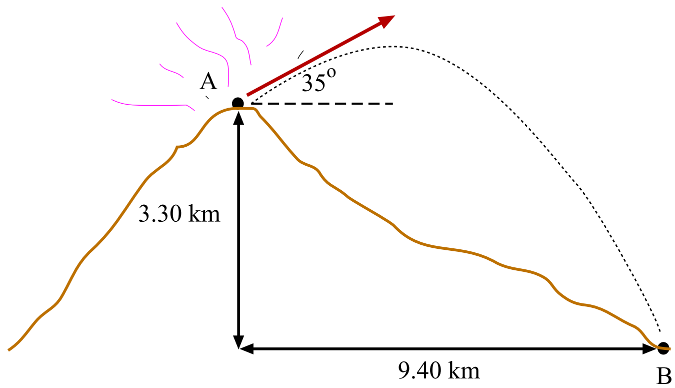

Figure 3.6: Volcanic bombs away!

The problem tells us that $R = 3H$. Substituting the expressions for H and R that we found in the last example (we pick the first expression we got for R), we get:

$$R = \frac{2v_0^2 \sin\theta_0 \cos\theta_0}{g} = 3H = 3\left(\frac{v_0^2 \sin^2\theta_0}{2g}\right)$$

Cancelling stuff, we get:

$$2\cos\theta_0 = \frac{3}{2}\sin\theta_0 \qquad \Longrightarrow \qquad \tan\theta_0 = \frac{4}{3}$$

The solution is:

$$\theta_0 = \tan^{-1}(4/3) = 53.1^\circ$$

The projectile was fired at 53.1° above the horizontal.

---

11. During volcanic eruptions, chunks of solid rock can be blasted out of a volcano; these projectiles are called *volcanic bombs*. Fig. 3.6 shows a cross section of Mt. Fuji in Japan. (a) At what initial speed would the bomb have to be ejected, at 35° to the horizontal, from the vent at A in order to fall at the foot of the volcano at B? (Ignore the effects of air on the bomb's travel.) (b) What would be the time of flight? [HRW5 4-42]

(a) We use a coordinate system with its origin at point A (the volcano "vent"); then for the flight from the vent at A to point B, the initial coordinates are $x_0 = 0$ and $y_0 = 0$, and the final coordinates are $x = 9.40\,\text{km}$ and $y = -3.30\,\text{km}$. Aside from this, we don't know the initial speed of the rock (that's what we're trying to find) or the time of flight from A to B. Of course, the acceleration of the rock is given by $a_x = 0$, $a_y = -g$.

We start with the x equation of motion. The initial $x-$velocity is

$$v_{0x} = v_0 \cos\theta$$

where $\theta = 35^\circ$ so the function $x(t)$ is

$$\begin{aligned} x &= x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2 \\ &= 0 + v_0\cos\theta t + 0 \\ &= v_0\cos\theta\, t \end{aligned}$$

Now, we *do* know that at the time of impact x had the value $x = 9.40\,\mathrm{km}$ so if we now let t be the time of flight, then

$$(9.40\,\mathrm{km}) = v_0\cos\theta t \qquad \text{or} \qquad t = \frac{(9.40\,\mathrm{km})}{v_0\cos\theta} \tag{3.25}$$

Next we look at the y equation of motion. Since $v_{0y} = v_0\sin\theta$ we get:

$$\begin{aligned} y &= y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2 \\ &= 0 + v_0\sin\theta t - \tfrac{1}{2}gt^2 \\ &= v_0\sin\theta\, t - \tfrac{1}{2}gt^2 \end{aligned}$$

But at the time t of impact the y coordinate had the value $y = -3.30\,\mathrm{km}$. If we also substitute for t in this expression using Eq. 3.25 we get:

$$\begin{aligned} -3.30\,\mathrm{km} &= v_0\sin\theta\left(\frac{9.40\,\mathrm{km}}{v_0\cos\theta}\right) - \tfrac{1}{2}g\left(\frac{9.40\,\mathrm{km}}{v_0\cos\theta}\right)^2 \\ &= (9.40\,\mathrm{km})\tan\theta - \frac{g(9.40\,\mathrm{km})^2}{2v_0^2\cos^2\theta} \end{aligned}$$

At this point we are done with the *physics* problem. The *only* unknown in this equation is v_0, which we can find by doing a little algebra:

$$\begin{aligned} \frac{g(9.40\,\mathrm{km})^2}{2v_0^2\cos^2\theta} &= (9.40\,\mathrm{km})\tan\theta + 3.30\,\mathrm{km} \\ &= 9.88\,\mathrm{km} \end{aligned}$$

which gives:

$$\begin{aligned} v_0^2 &= \frac{g(9.40\,\mathrm{km})^2}{\cos^2\theta(9.88\,\mathrm{km})} \\ &= \frac{g(0.951\,\mathrm{km})}{\cos^2\theta} \\ &= \frac{(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})(951\,\mathrm{m})}{\cos^2 35^\circ} \\ &= 1.39\times 10^4\,\frac{\mathrm{m}^2}{\mathrm{s}^2} \end{aligned}$$

and finally

$$v_0 = 118\,\frac{\mathrm{m}}{\mathrm{s}}$$

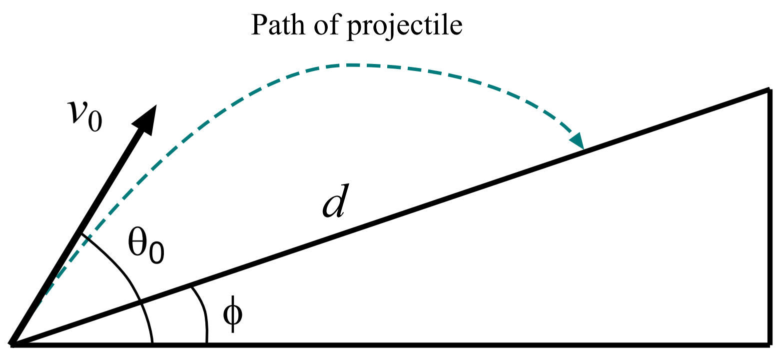

Figure 3.7: Projectile is fired up an incline, as described in Example 12

(b) Having v_0 in hand, finding t is easy. Using our result from part(a) and Eq. 3.25 we find:

$$t = \frac{(9.40\,\mathrm{km})}{v_0\cos\theta} = \frac{(9400\,\mathrm{m})}{(118\,\frac{\mathrm{m}}{\mathrm{s}})\cos 35^\circ} = 97.2\,\mathrm{s}$$

The time of flight is 97.2 s.

---

12. A projectile is fired up an incline (incline angle ϕ) with an initial speed v_0 at an angle θ_0 with respect to the horizontal ($\theta_0 > \phi$) as shown in Fig. 3.7. (a) Show that the projectile travels a distance d up the incline, where

$$d = \frac{2v_0^2\cos\theta_0\sin(\theta_0-\phi)}{g\cos^2\phi}$$

(b) For what value of θ_0 is d a maximum, and what is the maximum value? [Ser4 4-56]

(a) This is a relatively challenging problem, and of course it is completely analytic.

We can start by writing down equations for x and y as functions of time. By now we can easily see that we have:

$$x = v_0\cos\theta_0\,t$$
$$y = v_0\sin\theta_0\,t - \tfrac{1}{2}gt^2$$

We can combine these equations to get a relation between x and y for points on the trajectory; from the first, we have $t = x/(v_0\cos\theta_0)$, and putting this into the second one gives:

$$\begin{aligned} y &= v_0\sin\theta_0\left(\frac{x}{v_0\cos\theta_0}\right) - \tfrac{1}{2}g\left(\frac{x}{v_0\cos\theta_0}\right)^2 \\ &= (\tan\theta_0)x - \frac{g}{2}\frac{x^2}{v_0^2\cos^2\theta_0} \end{aligned}$$

What is the condition for the time that the projectile hits the slope? Unlike the problems where a projectile impacts with the flat ground or a wall, we don't know the value of x or y

at impact. But since the incline has a slope of $\tan\phi$, the relation between x and y for points on the slope is

$$y = (\tan\phi)x \ .$$

These two relations between x and y allow us to solve for the values of x and y where the impact occurs. Substituting for y above, we find:

$$(\tan\phi)x = (\tan\theta_0)x - \frac{g}{2}\frac{x^2}{v_0^2\cos^2\theta_0}$$

A little rearranging gives:

$$\frac{g}{2}\frac{x}{v_0^2\cos^2\theta_0} + (\tan\phi - \tan\theta_0) = 0$$

and the solution for x is:

$$x = \frac{2v_0^2\cos^2\theta_0(\tan\theta_0 - \tan\phi)}{g}$$

The problem has us solve for the distance d *up the slope*; this distance is related to the impact value of x by:

$$d = \frac{x}{\cos\phi}$$

and this gives us:

$$d = \frac{x}{\cos\phi} = \frac{2v_0^2\cos^2\theta_0(\tan\theta_0 - \tan\phi)}{g\cos\phi} \ .$$

Although this is a perfectly good expression for d, it is not the one presented in the problem. (Among other things, it has another factor of $\cos\phi$ downstairs.) If we multiply top and bottom by $\cos\phi$ we find:

$$\begin{aligned} d &= \frac{2v_0^2\cos^2\theta_0\cos\phi(\tan\theta_0 - \tan\phi)}{g\cos^2\phi} \\ &= \frac{2v_0^2\cos\theta_0(\cos\theta_0\cos\phi\tan\theta_0 - \cos\theta_0\cos\phi\tan\phi)}{g\cos^2\phi} \\ &= \frac{2v_0^2\cos\theta_0(\cos\phi\sin\theta_0 - \cos\theta_0\sin\phi)}{g\cos^2\phi} \end{aligned}$$

And now using an angle–addition identity from trigonometry in the numerator, we arrive at

$$d = \frac{2v_0^2\cos\theta_0\sin(\theta_0 - \phi)}{g\cos^2\phi}$$

which is the preferred expression for d.

(b) In part (a) we found the up–slope impact distance as a function of launch angle θ_0. (The launch speed v_0 and the slope angle ϕ are taken to be fixed.) For a certain value of $theta_0$, this function $d(\theta_0)$ will take on a maximum value. To find this value, we differentiate the function $d(\theta_0)$ and set the derivative equal to zero. We find:

$$\begin{aligned} d'(\theta_0) &= \frac{2v_0^2}{g\cos^2\phi}\frac{d}{d\theta_0}[\cos\theta_0\sin(\theta_0-\phi)] \\ &= \frac{2v_0^2}{g\cos^2\phi}[-\sin\theta_0\sin(\theta_0-\phi)+\cos\theta_0\cos(\theta_0-\phi)] \\ &= \frac{2v_0^2}{g\cos^2\phi}\cos(2\theta_0-\phi) \end{aligned}$$

where in the last step we used the trig identity $\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta)$.

Now, to satisfy $d'(\theta_0) = 0$ we must have $\cos(2\theta_0 - \phi) = 0$. While this equation has infinitely many solutions for θ_0, considering the values that θ_0 and ϕ may take on, we see that we need only look at the case where

$$2\theta_0 - \phi = \frac{\pi}{2}$$

which of course, *does* solve the equation. This gives us:

$$\theta_0 = \frac{\pi}{4} + \frac{\phi}{2}$$

for the value of θ which makes the projectile go the farthest distance d up the slope.

To find what this value of d is, we substitute for θ_0 in our function $d(\theta_0)$. We find:

$$\begin{aligned} d_{\rm max} &= \frac{2v_0^2}{g\cos^2\phi}\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\sin\left(\frac{\pi}{4}+\frac{\phi}{2}-\phi\right) \\ &= \frac{2v_0^2}{g\cos^2\phi}\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right) \end{aligned}$$

This expression is *correct* but it can be simplified. We use the trig identity which states:

$$\sin\alpha\cos\beta = \tfrac{1}{2}\sin(\alpha+\beta) + \tfrac{1}{2}\sin(\alpha-\beta)$$

this gives us:

$$\begin{aligned} \sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right) &= \tfrac{1}{2}\sin\left(\frac{\pi}{2}\right) + \tfrac{1}{2}\sin(-\phi) \\ &= \tfrac{1}{2} - \tfrac{1}{2}\sin\phi \\ &= \tfrac{1}{2}(1-\sin\phi) \end{aligned}$$

which *is* a lot simpler. Using this result in our expression for $d_{\rm max}$ gives:

$$d_{\rm max} = \frac{2v_0^2}{g\cos^2\phi}\frac{(1-\sin\phi)}{2} = \frac{v_0^2(1-\sin\phi)}{g(1-\sin^2\phi)} = \frac{v_0^2}{g(1+\sin\phi)}$$

which is a simple as it's going to get!

We can *check* result for a couple well–known cases. If $\phi = 0$ we are dealing with the common projectile problem on level ground for which we know we get maximum range when $\theta_0 = 45^\circ$ and from our solution for that problem we get $R = \frac{v_0^2}{g}$. If $\phi = 90^\circ$ we have the problem of a projectile fired straight up; one can show that the maximum height reached is $H = \frac{v_0^2}{2g}$ which again agrees with the formula we've derived.

3.2.5 Uniform Circular Motion

---

13. In one model of the hydrogen atom, an electron orbits a proton in a circle of radius $5.28 \times 10^{-11}\,\mathrm{m}$ with a speed of $2.18 \times 10^{6}\,\frac{\mathrm{m}}{\mathrm{s}}$. (a) What is the acceleration of the electron in this model? (b) What is the period of the motion? [HRW5 4-57]

(a) The electron moves in a circle with constant speed. It is accelerating *toward the center of the circle* and the acceleration has magnitude $a_{\mathrm{cent}} = \frac{v^2}{r}$. Substituting the given values, we have:

$$a_{\mathrm{cent}} = \frac{v^2}{r} = \frac{(2.18 \times 10^{6}\,\frac{\mathrm{m}}{\mathrm{s}})^2}{(5.28 \times 10^{-11}\,\mathrm{m})} = 9.00 \times 10^{22}\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

The acceleration has magnitude $9.00 \times 10^{22}\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

(b) As the electron makes one trip around the circle of radius r, it moves a distance $2\pi r$ (the circumference of the circle). If T is the period of the motion, then the speed of the electron is given by the ratio of distance to time,

$$v = \frac{2\pi r}{T} \qquad \text{which gives...} \qquad T = \frac{2\pi r}{v}$$

which shows why Eq. 3.22 is true. Substituting the given values, we get:

$$T = \frac{2\pi(5.28 \times 10^{-11}\,\mathrm{m})}{(2.18 \times 10^{6}\,\frac{\mathrm{m}}{\mathrm{s}})} = 1.52 \times 10^{-16}\,\mathrm{s}$$

The period of the electron's motion is $1.52 \times 10^{-16}\,\mathrm{s}$.

---

14. A rotating fan completes 1200 revolutions every minute. Consider a point on the tip of a blade, at a radius of $0.15\,\mathrm{m}$. (a) Through what distance does the point move in one revolution? (b) What is the speed of the point? (c) What is its acceleration? (d) What is the period of the motion? [HRW5 4-63]

(a) As the fan makes one revolution, the point in question moves through a circle of radius $0.15\,\mathrm{m}$ so the distance it travels is the circumference of that circle, i.e.

$$d = 2\pi r = 2\pi(0.15\,\mathrm{m}) = 0.94\,\mathrm{m}$$

The point travels $0.94\,\mathrm{m}$.

(b) If in one minute ($60\,\mathrm{s}$) the fan makes 1200 revolutions, the time to make *one* revolution must be

$$\text{Time for one rev} = T = \frac{1}{1200} \cdot (1.00\,\mathrm{min}) = \frac{1}{1200} \cdot (60.0\,\mathrm{s}) = 5.00 \times 10^{-2}\,\mathrm{s}$$

Using our answer from part (a), we know that the point travels $0.94\,\mathrm{m}$ in $5.000 \times 10^{-2}\,\mathrm{s}$, moving at constant speed. Therefore that speed is:

$$v = \frac{d}{T} = \frac{0.94\,\mathrm{m}}{5.000 \times 10^{-2}\,\mathrm{s}} = 19\,\frac{\mathrm{m}}{\mathrm{s}}$$

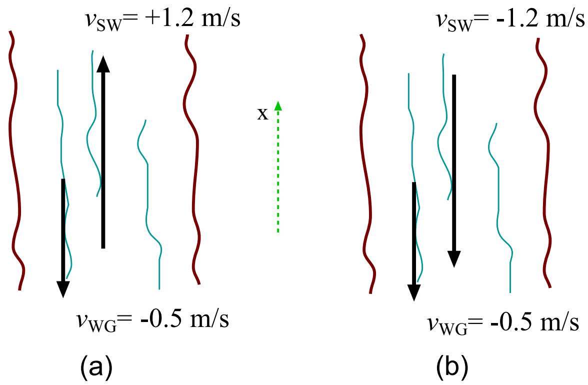

Figure 3.8: (a) Velocities for case where swimmer swims upstream. (b) Velocities for case where swimmer swims downstream.

(c) The point is undergoing uniform circular motion; its acceleration is always *toward the center* and has magnitude $a_{\rm cent} = \frac{v^2}{r}$. Substituting,

$$a_{\rm cent} = \frac{v^2}{r} = \frac{(19\,\frac{\rm m}{\rm s})^2}{(0.15\,{\rm m})} = 2.4 \times 10^3\,\frac{\rm m}{\rm s^2}$$

(d) The period of the motion is the time for the fan to make one revolution. And we already found this in part (b)! It is:

$$T = 5.00 \times 10^{-2}\,{\rm s}$$

3.2.6 Relative Motion

---

15. A river has a steady speed of $0.500\,\frac{\rm m}{\rm s}$. A student swims upstream a distance of $1.00\,{\rm km}$ and returns to the starting point. If the student can swim at a speed of $1.20\,\frac{\rm m}{\rm s}$ in still water, how long does the trip take? Compare this with the time the trip would take if the water were still. [Ser4 4-43]

What happens if the water *is* still? The student swims a distance of 1.00 km "upstream" at a speed of $1.20\,\frac{\rm m}{\rm s}$; using the simple distance/time formula $d = vt$ the time for the trip is

$$t = \frac{d}{v} = \frac{1.0 \times 10^3\,{\rm m}}{1.20\,\frac{\rm m}{\rm s}} = 833\,{\rm s}$$

and the same is true for the trip back "downstream". So the total time for the trip is

$$833\,{\rm s} + 833\,{\rm s} = 1.67 \times 10^3\,{\rm s} = 27.8\,{\rm min}$$

Good enough, but what about the case where the water is *not* still? And what does that have to do with relative velocities? In Fig. 3.8, the river is shown; it flows in the $-x$

direction. At all times, the velocity of the water *with respect to the ground* is

$$v_{\rm WG} = -0.500\,\tfrac{\rm m}{\rm s}\ .$$

When the student swims upstream, as represented in Fig. 3.8(a), his velocity *with respect to the water* is

$$v_{\rm SW} = +1.20\,\tfrac{\rm m}{\rm s}\ .$$

We know this because we are given his swimming speed for *still* water.

Now we are interested in the student's velocity *with respect to the ground*, which we will call $v_{\rm SG}$. It is given by the sum of his velocity with respect to the water and the water's velocity with respect to the ground:

$$v_{\rm SG} = v_{\rm SW} + v_{\rm WG} = +1.20\,\tfrac{\rm m}{\rm s} - 0.500\,\tfrac{\rm m}{\rm s} = 0.70\,\tfrac{\rm m}{\rm s}$$

and so to cover a displacement of $\Delta x = 1.00\,{\rm km}$ (measured along the ground!) requires a time

$$\Delta t = \frac{\Delta x}{v_{\rm SG}} = \frac{1.00\times 10^3\,{\rm m}}{0.70\,\frac{\rm m}{\rm s}} = 1.43\times 10^3\,{\rm s}$$

Then the student swims downstream (Fig. 3.8(b)) and his velocity with respect to the water is

$$v_{\rm SW} = -1.20\,\tfrac{\rm m}{\rm s}$$

giving him a velocity with respect to the ground of

$$v_{\rm SG} = v_{\rm SW} + v_{\rm WG} = -1.20\,\tfrac{\rm m}{\rm s} - 0.500\,\tfrac{\rm m}{\rm s} = 1.70\,\tfrac{\rm m}{\rm s}$$

so that the time to cover a displacement of $\Delta x = -1.00\,{\rm km}$ is

$$\Delta t = \frac{\Delta x}{v_{\rm SG}} = \frac{(-1.00\times 10^3\,{\rm m})}{(-1.70\,\frac{\rm m}{\rm s})} = 5.88\times 10^2\,{\rm s}$$

The *total* time to swim upstream and then downstream is

$$\begin{aligned} t_{\rm Total} &= t_{\rm up} + t_{\rm down} \\ &= 1.43\times 10^3\,{\rm s} + 5.88\times 10^2\,{\rm s} = 2.02\times 10^3\,{\rm s} = 33.6\,{\rm min}\ . \end{aligned}$$

---

16. A light plane attains an airspeed of $500\,{\rm km/hr}$. The pilot sets out for a destination $800\,{\rm km}$ to the north but discovers that the plane must be headed 20.0° east of north to fly there directly. The plane arrives in $2.00\,{\rm hr}$. What was the wind velocity vector? [HRW5 4-83]

Whoa! What the Hell is this problem talking about???

When a plane flies in air which *itself* is moving (i.e. there is a wind velocity) there are three (vector) velocities we need to think about; I will refer to them as:

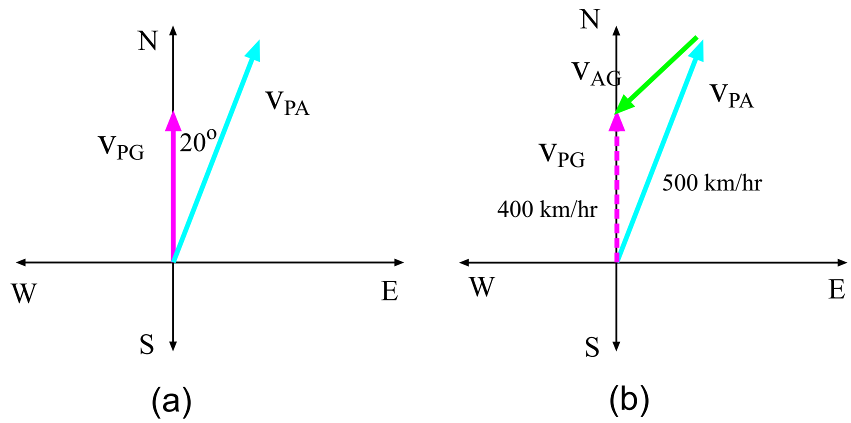

Figure 3.9: (a) Vectors for the planes velocity with respect to the ground ($\mathbf{v}_{\rm PG}$) and with respect to the moving air ($\mathbf{v}_{\rm PA}$). (b) The sum of the plane's velocity relative to the air and the wind velocity gives the plane's velocity with respect to the ground, $\mathbf{v}_{\rm PG}$.

$\mathbf{v}_{\rm PA}$: Velocity of the plane with respect to the air. The magnitude of this vector is the "airspeed" of the plane. (This is the only thing that a plane's "speedometer" can really measure.)

$\mathbf{v}_{\rm AG}$: Velocity of the air with respect to the ground. This is the wind velocity.

$\mathbf{v}_{\rm PG}$: Velocity of the plane with respect to the ground. This is the quantity which tells us the rate of (ground!) travel of the plane.

These three vectors are related via:

$$\mathbf{v}_{\rm PG} = \mathbf{v}_{\rm PA} + \mathbf{v}_{\rm AG}$$

The first thing we are given in this problem is that the magnitude of $\mathbf{v}_{\rm PA}$ is $500\,{\rm km/\,hr}$. The plane needs to fly *due north* and this tells us that $\mathbf{v}_{\rm PG}$ (the real direction of motion of the plane) points north (along the y axis). We are then told that the plane's "heading" is 20.0° east of north, which tells us that the direction of $\mathbf{v}_{\rm PA}$ lies in this direction. These facts are illustrated in Fig. 3.9(a).

Now if the plane travels $800\,{\rm km}$ in $2.00\,{\rm hr}$ then its speed (with respect to the ground!) is

$$v_{\rm PG} = \frac{800\,{\rm km}}{2.00\,{\rm hr}} = 400\,\frac{\rm km}{\rm hr}\ .$$

which we note in Fig. 3.9(b). Since we now have the magnitudes and and directions of $\mathbf{v}_{\rm PA}$ and $\mathbf{v}_{\rm PG}$ we can compute the wind velocity,

$$\mathbf{v}_{\rm AG} = \mathbf{v}_{\rm PG} - \mathbf{v}_{\rm PA}$$

The x component of this vector is

$$\mathbf{v}_{{\rm AG},x} = 0 - 500\,\tfrac{\rm km}{\rm hr}\sin 20.0^\circ = -171\,\tfrac{\rm km}{\rm hr}$$

and its y component is

$$\mathbf{v}_{{\rm AG},y} = 400 - 500\,\tfrac{\rm km}{\rm hr}\cos 20.0^\circ = -69.8\,\tfrac{\rm km}{\rm hr}$$

So the wind velocity is

$$\mathbf{v}_{AG} = -171\,\tfrac{\mathrm{km}}{\mathrm{hr}}\mathbf{i} - 69.8\,\tfrac{\mathrm{km}}{\mathrm{hr}}\mathbf{j}$$

If we want to express the velocity as a magnitude and direction, we find:

$$v_{\mathrm{AG}} = \sqrt{(-171\,\tfrac{\mathrm{km}}{\mathrm{hr}})^2 + (-69.8\,\tfrac{\mathrm{km}}{\mathrm{hr}})^2} = 185\,\tfrac{\mathrm{km}}{\mathrm{hr}}$$

so the wind speed is $185\,\frac{\mathrm{km}}{\mathrm{hr}}$. The direction of the wind, measured as an angle θ counter-clockwise from the east is found from its components:

$$\tan\theta = \frac{-69.8}{-171} = 0.408 \qquad \Longrightarrow \qquad \theta = \tan^{-1}(0.408) = 202^\circ$$

(Here we have made sure to get the angle right! Since both components are negative, θ lies in the third quadrant!) Since 180° would be *due West* and the wind direction is 22° larger than that, we can also say that the wind direction is "22° south of west".